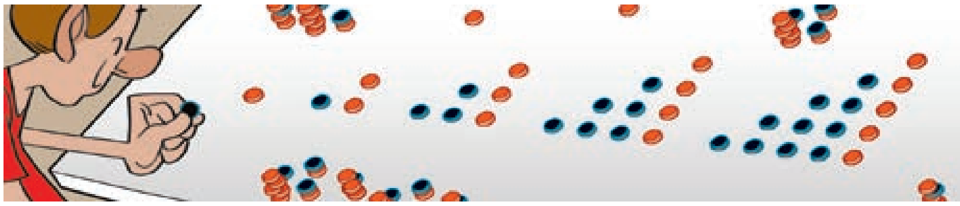
1 Sequences

Pablo is playing with some counters.



He constructed triangles first by placing one counter and then adding one column at a time, each with one more counter. The number of counters he uses each time is:

1 $\quad 1+2=3 \quad 3+3=6 \quad 6+4=10 \quad 10+5=15$

The number of counters he uses for each figure forms a **sequence**.

A **sequence** is an organised list of real numbers: a: a_1, a_2, a_3, a_4, a_5...

Each number in a sequence is called a **term**. The **index** of each term indicates the term's position in the sequence.

How many counters will Pablo need for the next figure? We can calculate this by finding the sum of the counters that correspond to the position in the sequence: $a_6 = 15 + 6 = 21$ counters.

This way of defining a sequence is called **by recurrence**, where a term is obtained from the previous one. In this case:

$$a_1 = 1,\ a_n = a_{n-1} + n \text{ for } n > 1$$

A **sequence** is **recurring** if we can construct terms with an algebraic expression, called the **law of recurrence**, from the previous terms.

Is it possible to find out how many counters Pablo needs for figure number 20 without having to calculate all the previous terms?

By manipulating the triangles, he notices that two equal triangles create a rectangle. This will help him work out how many counters he needs. He organises all the data into a table and writes the relationship between the counters he needs and the position that the figure occupies.

Position	1	2	3	4	...	n
Number of counters	$\frac{1\cdot 2}{2}=1$	$\frac{2\cdot 3}{2}=3$	$\frac{3\cdot 4}{2}=6$	$\frac{4\cdot 5}{2}=10$	...	$\frac{n\cdot(n+1)}{2}$

The ***n*th term** for the sequence is: $a_n = \dfrac{n\cdot(n+1)}{2}$

So, for position number 20, Pablo will need $a_{20} = \dfrac{20\cdot 21}{2} = 210$ counters.

The ***n*th term** of a sequence is an algebraic expression that helps calculate any term in the sequence once we know its position.

Mathematical language

- Sequences are named using lower case letters: a, b, c...
- Terms are named by indicating the name of the sequence and its position:

  a sub-one $\to a_1$

  a sub-two $\to a_2$

  a sub-three $\to a_3$

  ...

  a sub-n $\to a_n$
- **Interpolation** of terms in a sequence consists of obtaining the terms that are between two given terms.

Activities

1 In your notebook, write the terms indicated in these sequences.

a) a_1, a_3 and a_5 in the sequence a:
5, 15, 20, 25, 30, 35...

b) b_2, b_4 and b_7 in the sequence b:
1, −2, 3, −4, 5, −6, 7...

c) c_1, c_4 and c_6 in the sequence c:
$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}...$

d) d_3, d_5 and d_6 in the sequence d
2, 5, 7, 12, 19, 31, 50...

2 Write the next four terms of these sequences. Explain how you calculated them.

a) 7, 5, 3, 1, −1...

b) 1,1; 1,01; 1,001; 1,0001; 1,00001...

c) 1, 4, 9, 16, 25...

d) $\frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{7}{16}, \frac{9}{32}...$

3 Write the first five terms of the sequences given by these *n*th terms.

a) $a_n = 5 - n$

b) $b_n = \frac{3}{n}$

c) $c_n = n^2 - 1$

d) $d_n = (-2)^n$

4 Calculate the fifth, tenth and hundredth terms in these sequences using the *n*th term.

a) $a_n = 5n + 1$ **b)** $b_n = n - \frac{2}{n}$ **c)** $c_n = \frac{n+2}{3-n}$

5 For each of these sequences, write the *n*th term and then find the tenth term.

a) 17, 13, 9, 5, 1, −3...

b) 0, 1, 8, 27, 64, 125...

c) 8; 4; 2; 1; 0,5; 0,25...

d) $1, \frac{3}{4}, \frac{2}{3}, \frac{5}{8}, \frac{3}{5}, \frac{7}{12}...$

6 Write first five terms of the sequence defined by recurrence: *the first term is 3 and every other term is double the previous term, minus one.*

7 These sequences are recurrent. Calculate the next five terms for each.

a) $a_1 = 4$, $a_n = a_{n-1} + 5$ for $n > 1$

b) $b_1 = 1$, $b_n = 2b_{n-1} - 3$ for $n > 1$

c) $c_1 = 1$, $c_2 = 3$, $c_n = 2c_{n-2} - c_{n-1}$ for $n > 2$

d) $d_1 = 2$, $d_2 = -1$, $d_3 = 1$, $d_n = d_{n-3} + d_{n-1}$ for $n > 3$

8 Find the next three terms in these sequences and write their laws of recurrence.

a) −5, −3, 0, 4, 9, 15, 22...

b) 3, 4, 1, −3, −4, −1, 3...

c) 1, 3, 5, 11, 21, 43, 85...

d) 1, 2, 3, 6, 11, 20, 37...

9 Observe the way the terms act in the sequence $a_n = (-1)^n$. Use it to write the *n*th term of this other sequence: 1, −1, 1, −1, 1, −1, 1...

10 Write the *n*th term of these sequences.

a) −3, 9, −27, 81, −243, 729...

b) 0,1; −0,01; 0,001; −0,0001; 0,00001...

c) −1, 2, −3, 4, −5, 6...

d) 2, −4, 6, −8, 10, −12...

11 In his book *Liber abaci* the mathematician Leonardo de Pisa, Fibonacci, poses this problem: *how many pairs of rabbits will reproduce in a year, starting with one male–female pair, if each month each male–female pair produces another male–female pair, who also reproduce the following month?*

a) Write the number of pairs of rabbits there will be each month for twelve months. These are the first twelve terms of the Fibonacci sequence.

b) Analyse the results and explain how you found them. Look for a law of recurrence that describes the sequence.

c) Use your calculator to check that the sequence also adjusts to the *n*th term.

Binet's formula: $f_n = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{\sqrt{5} \cdot 2^n}$

CLIL zone

12 Listen to the documentary about bees. Draw a diagram to show the sequence of a drone's family tree, indicating the number of:

a) parents. **b)** grandparents. **c)** great-grandparents.