

OXFORD IB DIPLOMA PROGRAMME



MATHEMATICS: APPLICATIONS AND INTERPRETATION

HIGHER LEVEL
COURSE COMPANION

 ENHANCED ONLINE

Suzanne Doering
Panayiotis Economopoulos
Peter Gray
David Harris

Tony Halsey
Michael Ortman
Nuriye Sirinoglu Singh
Jennifer Chang Wathall

OXFORD

Contents

Introduction	vii	4 Modelling constant rates of change: linear functions and regressions	140
How to use your enhanced online course book	ix	4.1 Functions	142
1 Measuring space: accuracy and geometry	2	4.2 Linear models.....	155
1.1 Representing numbers exactly and approximately	4	4.3 Inverse functions	168
1.2 Angles and triangles	13	4.4 Arithmetic sequences and series	178
1.3 Three dimensional geometry.....	28	4.5 Linear regression	190
Chapter review	39	Chapter review.....	198
Modelling and investigation activity.....	42	Modelling and investigation activity.....	202
2 Representing and describing data: descriptive statistics	44	5 Quantifying uncertainty: probability	204
2.1 Collecting and organizing data	46	5.1 Reflecting on experiences in the world of chance. First steps in the quantification of probabilities	206
2.2 Statistical measures	51	5.2 Representing combined probabilities with diagrams	212
2.3 Ways in which you can present data	59	5.3 Representing combined probabilities with diagrams and formulae	216
2.4 Bivariate data.....	66	5.4 Complete, concise and consistent representations	221
Chapter review.....	75	Chapter review	226
Modelling and investigation activity.....	80	Modelling and investigation activity.....	230
3 Dividing up space: coordinate geometry, Voronoi diagrams, vectors, lines	82	6 Modelling relationships with functions: power and polynomial functions	232
3.1 Coordinate geometry in 2 and 3 dimensions	84	6.1 Quadratic models	234
3.2 The equation of a straight line in 2 dimensions	86	6.2 Problems involving quadratics	245
3.3 Voronoi diagrams	96	6.3 Cubic functions and models	259
3.4 Displacement vectors	104	6.4 Power functions, direct and inverse variation and models.....	268
3.5 The scalar and vector product.	112	Chapter review	281
3.6 Vector equations of lines	118	Modelling and investigation activity.....	286
Chapter review	130		
Modelling and investigation activity.....	134		
Paper 3 question and comments	136		

7 Modelling rates of change: exponential and logarithmic functions288

7.1 Geometric sequences and series.....	290
7.2 Financial applications of geometric sequences and series	302
7.3 Exponential functions and models	310
7.4 Laws of exponents – laws of logarithms.....	320
7.5 Logistic models	329
Chapter review	332
Modelling and investigation activity	336

8 Modelling periodic phenomena: trigonometric functions and complex numbers338

8.1 Measuring angles.....	340
8.2 Sinusoidal models: $f(x) = a \sin(b(x-c)) + d$	343
8.3 Completing our number system.....	352
8.4 A geometrical interpretation of complex numbers	356
8.5 Using complex numbers to understand periodic models.....	364
Chapter review	368
Modelling and investigation activity.....	370

9 Modelling with matrices: storing and analysing data372

9.1 Introduction to matrices and matrix operations	374
9.2 Matrix multiplication and Properties	377
9.3 Solving systems of equations using matrices	383
9.4 Transformations of the plane.....	391
9.5 Representing systems	402
9.6 Representing steady state systems.....	406
9.7 Eigenvalues and eigenvectors.....	413
Chapter review	422
Modelling and investigation activity.....	424

10 Analyzing rates of change: differential calculus426

10.1 Limits and derivatives	428
10.2 Differentiation: further rules and techniques.....	442
10.3 Applications and higher derivatives	455
Chapter review	467
Modelling and investigation activity.....	470

11 Approximating irregular spaces: integration and differential equations472

11.1 Finding approximate areas for irregular regions.....	474
11.2 Indefinite integrals and techniques of integration	487
11.3 Applications of integration	498
11.4 Differential equations	515
11.5 Slope fields and differential equations	519
Chapter review	526
Modelling and investigation activity.....	530

12 Modelling motion and change in two and three dimensions532

12.1 Vector quantities	535
12.2 Motion with variable velocity	540
12.3 Exact solutions of coupled differential equations	547
12.4 Approximate solutions to coupled linear equations	556
Chapter review	563
Modelling and investigation activity.....	568

13 Representing multiple outcomes: random variables and probability distributions570

13.1 Modelling random behaviour	572
13.2 Modelling the number of successes in a fixed number of trials	580

13.3 Modelling the number of successes in a fixed interval	587
13.4 Modelling measurements that are distributed randomly	593
13.5 Mean and variance of transformed or combined random variables	601
13.6 Distributions of combined random variables	604
Chapter review	616
Modelling and investigation activity.....	620

14 Testing for validity: Spearman's, hypothesis testing and χ^2 test for independence 622

14.1 Spearman's rank correlation Coefficient.....	625
14.2 Hypothesis testing for the binomial probability, the Poisson mean and the product moment correlation coefficient	629
14.3 Testing for the mean of a normal distribution	638
14.4 χ^2 test for independence	650
14.5 χ^2 goodness-of-fit test	657
14.6 Choice, validity and interpretation of tests	670
Chapter review	684
Modelling and investigation activity.....	688

15 Optimizing complex networks: graph theory 690

15.1 Constructing graphs	692
15.2 Graph theory for unweighted graphs	699
15.3 Graph theory for weighted graphs: the minimum spanning tree	708
15.4 Graph theory for weighted graphs: the Chinese postman problem.....	714

15.5 Graph theory for weighted graphs: the travelling salesman problem.....	720
Chapter review	729
Modelling and investigation activity.....	734

16 Exploration 736

Practice exam paper 1 750

Practice exam paper 2 756

Practice exam paper 3 761

Answers 765

Index 849

Digital contents



Digital content overview

Click on this icon here to see a list of all the digital resources in your enhanced online course book. To learn more about the different digital resource types included in each of the chapters and how to get the most out of your enhanced online course book, go to page ix.



Syllabus coverage

This book covers all the content of the Mathematics: applications and interpretation HL course. Click on this icon here for a document showing you the syllabus statements covered in each chapter.



Practice exam papers

Click on this icon here for an additional set of practice exam papers.



Worked solutions

Click on this icon here for worked solutions for all the questions in the book.

1 Measuring space: accuracy and geometry

In this chapter you will approximate and calculate measures of angles, distances, areas and volumes in two and three dimensions. You will determine how these tools are used in astronomy to measure the distance to nearby stars, in cartography to measure distances between landmarks, and in navigation to determine the course of ships and aircrafts. In addition, you will explore other applications in surveying, architecture, disaster assessment and biology.

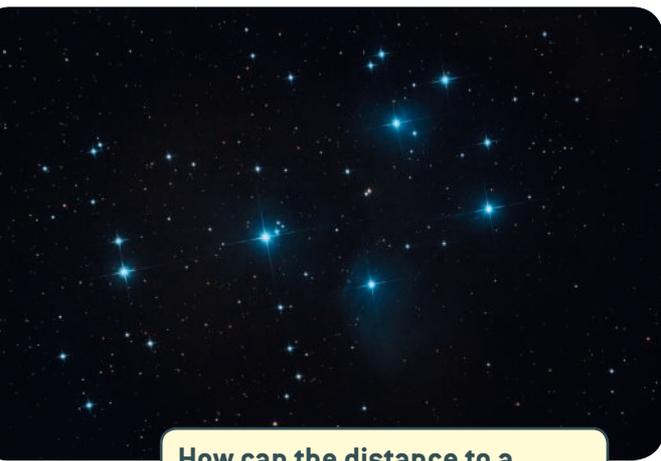
Concepts

- Space
- Quantity

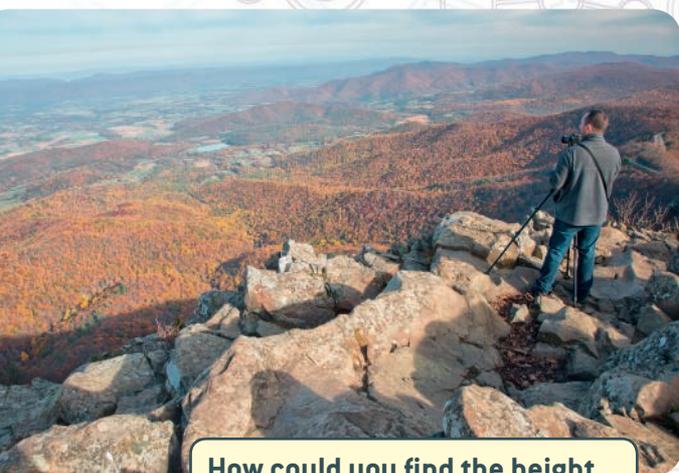


Microconcepts

- Rounding
- Percentage error
- Standard form
- Operations with rational exponents
- Right triangles: Sine, cosine, tangent ratios
- Angles of elevation and depression
- Bearings
- Non-right triangles: Sine rule and cosine rule
- Circles: Length of arc, area of sector
- Volume and surface area of 3D figures
- Angle between line and plane



How can the distance to a nearby star be measured?

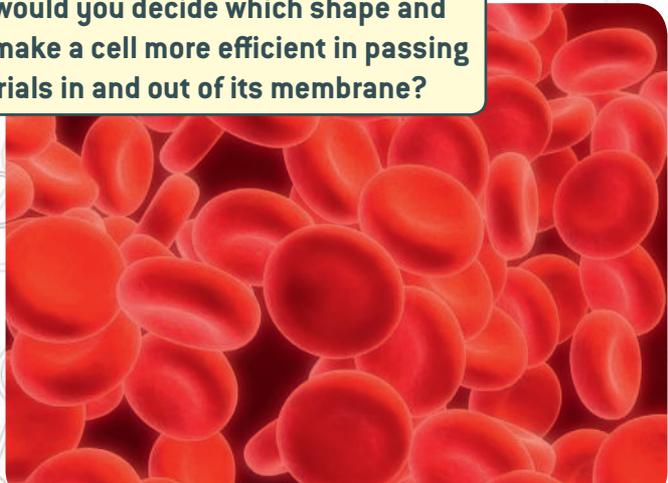


How could you find the height of a mountain peak?

How does a sailor calculate the distance to the coast?



How would you decide which shape and size make a cell more efficient in passing materials in and out of its membrane?



In 1856 The Great Trigonometric Survey of India measured the height of Mount Everest, known in Nepali as Sagarmatha and in Tibetan as Chomolungma. The surveyors measured the distance between two points at sea level and then measured the angles between the top of the mountain and each point.

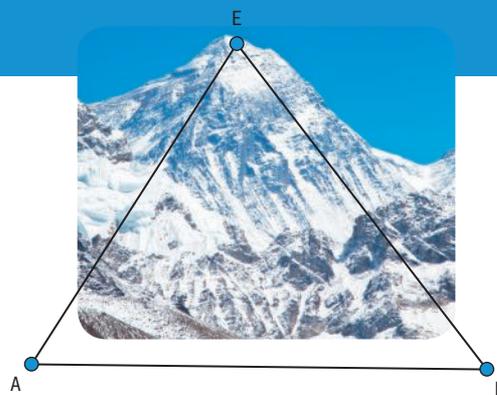
The summit of Mount Everest is labeled E, the two points A and B are roughly at sea level and are 33 km apart.

If $\hat{E} = 90^\circ$, what would be the maximum possible height of Mount Everest?

Think about and then **write down** your own intuitive answer to each question.

Discuss your answer with a friend then **share** your ideas with your class.

- Why do you think it took so long to determine the elevation of Mount Everest?
- Why do you think surveyors prefer to measure angles, but not lengths of sides?
- What assumptions are made?



Developing inquiry skills

Write down any similar inquiry questions you might ask if you were asked to find the height of tree, the distance between two towns or the distance between two stars.

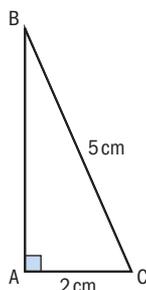
What questions might you need to ask in these scenarios which differ from the scenario where you are estimating the height of Mount Everest?

Think about the questions in this opening problem and answer any you can. As you work through the chapter, you will gain mathematical knowledge and skills that will help you to answer them all.

Before you start

You should know how to:

- 1 Round to significant figures and decimal places.
eg Write the number 80.426579 to:
 - a 2 decimal places
 - b 3 significant figures.
- 2 Evaluate integer and rational exponents.
eg $3^{-4} = \frac{1}{3^4} = \frac{1}{81}$, $16^{\frac{1}{2}} = \sqrt{16} = 4$
- 3 Use properties of triangles, including Pythagoras' theorem.



eg Find the length of AB in this triangle.

$$AB^2 = BC^2 - AC^2$$

$$AB^2 = 5^2 - 2^2$$

$$AB^2 = 21$$

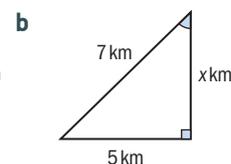
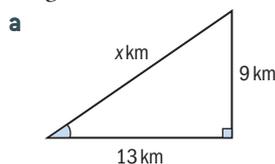
$$AB = \sqrt{21} = 4.58 \text{ cm (3 s.f.)}$$

Skills check

[Click here for help with this skills check](#)



- 1 Write down each number rounded to:
 - a 2 decimal places
 - b 3 significant figures.
 - i 0.6942
 - ii 28.706
 - iii 77.984561
- 2 Evaluate
 - a 2^{-3}
 - b $27^{\frac{1}{3}}$
- 3 In each of these right triangles, find the length of side x .



1.1 Representing numbers exactly and approximately

In mathematics and in everyday life, we frequently encounter numbers that have been measured or estimated. In this section you will investigate how we can quantify uncertainty in the numbers and calculations we use throughout this course.

Investigation 1

Margaret Hamilton worked for NASA as the lead developer for Apollo flight software. The photo here shows her in 1969, standing next to the books of navigation software code that she and her team produced for the Apollo mission that first sent humans to the Moon.

- 1 Estimate the height of the books of code stacked together, as shown in the image. What assumptions are you making?
- 2 Estimate the number of pages of code for the Apollo mission. How would you go about making this estimation? What assumptions are you making?
- 3 **Factual** What is an estimate? What is estimation? How would you go about estimating? How can comparing measures help you estimate?
- 4 **Conceptual** Why are estimations useful?



Recall that we can specify accuracy using **significant figures (digits)**. The digits that can be determined accurately are called **significant figures**. Thus, a scale that could only register up to the hundredths of a gram mass until 99.99 g, could only measure up to 4 figures of accuracy (4 significant figures).

In your IA or on specific exam questions, you may need to choose an appropriate degree of accuracy. When performing operations on measurements, round answers to the same number of significant figures as the least accurate measurement. This is illustrated in the example below.

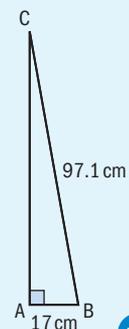
EXAM HINT

Write your final answers as exact values or rounded to at least 3 s.f., unless otherwise instructed. Round only your final answer and not any intermediate calculations.

Example 1

A component of an aircraft wing is being designed in the shape of a right triangle. One of the legs must measure 17 cm and the hypotenuse must measure 97.1 cm, as shown in the diagram.

- a Find the height of the triangle, rounding your answer **to the given degree of accuracy**.
 - i To 4 d.p. (decimal places)
 - ii To 2 s.f.
- b Find the area of the material (in cm^2) necessary to manufacture the component to an appropriate degree of accuracy.
- c Show that intermediate rounding to 2 s.f. leads to an inaccurate answer.





- a** i 95.6003 cm
ii 96 cm

$$\begin{aligned} \mathbf{b} \quad A &= \frac{1}{2}bh = \frac{1}{2} \times 17 \times 95.60026151\dots \\ &= 812.602\dots \\ &= 810 \text{ cm}^2 \text{ (2 s.f.)} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad A &= \frac{1}{2} \times 17 \times 96 = 816 \\ &= 820 \text{ cm}^2 \text{ (2 s.f.)} \\ &\neq 810 \text{ cm}^2 \end{aligned}$$

Use Pythagoras to find the height of the triangle: $\sqrt{97.1^2 - 17^2} = 95.60026151\dots$

If the question did not specify a level of accuracy, we would round to 3 s.f.: 95.6 cm.

Use the most accurate value (store or copy/paste with technology) when calculating. You could also write down your work

as $A = \frac{1}{2} \times 17 \times 95.6$. Do not use this to calculate!

Round area to 2 s.f. because the least accurate measurement, 17 cm, has 2 s.f.

This shows why rounding intermediate answers should be avoided in all calculations.

Exercise 1A

- A restaurant is remodelling and replacing its circular tables with square tables. They want the new tables to have the same area as the old ones. The circumference of the circular tables is measured to be 4.1 m. Find the side length of the new square tables
 - to an appropriate degree of accuracy
 - to 3 s.f.
- The heights of 10 koalas, measured to the nearest cm, are: 81, 73, 71, 80, 76, 84, 73, 88, 91, 75.
Find the mean (average) height of the koalas to an appropriate degree of accuracy.

Bounds and error

Suppose you find the weight of a bag of coffee as 541.5 g, accurate to the nearest 0.1 g. Then the exact weight w of the bag could be anywhere in the interval $541.45 \text{ g} \leq w < 541.55 \text{ g}$, as all of these values would round to 541.5 g.

If a measurement M is accurate to a particular unit u , then its exact value V lies in the interval

$$M - 0.5u \leq V < M + 0.5u$$

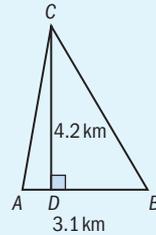
The endpoints of this interval are called the **lower** and **upper bounds**. Upper and lower bounds provide one way to quantify the **uncertainty** of a measurement when the exact value is unknown.

TOK

How does the perception of the language being used distort our understanding?

Example 2

A state park is created on a triangular area between roads. The triangular area is measured to have a base length of 3.1 km and corresponding height of 4.2 km. The measurement tool is accurate to the nearest tenth of a kilometre. Find the upper bound to the area of the park.



$$3.1 + 0.05 = 3.15 \text{ km}$$

$$4.2 + 0.05 = 4.25 \text{ km}$$

$$\text{Area of park} < \frac{1}{2} \times 3.15 \times 4.25$$

$$\text{Area of park} < 6.69 \text{ km}^2 \text{ (3 s.f.)}$$

The upper and lower bounds for the base length will be

$$3.1 - 0.05 \leq b < 3.1 + 0.05$$

$$3.05 \leq b < 3.15$$

and for the height

$$4.2 - 0.05 \leq h < 4.2 + 0.05$$

$$4.15 \leq h < 4.25$$

Using the area of triangle and upper bounds.

$$\text{Area of a triangle} = \frac{1}{2} \times \text{base} \times \text{height}$$

Since measurements are approximate there is always error in the measurement results. A measurement error is the difference between the exact value (V_E) and the approximate value (V_A), ie:

$$\text{Measurement error} = V_A - V_E$$

Investigation 2

Tomi and Massimo measured the length of a yardstick and the length of a foot and obtained 92.44 cm for the length of a yard and 31.48 cm for the length of a foot.

- Given that the exact values of 1 yard is 91.44 cm and 1 foot is 30.48 cm, find the measurement error in the two measurements obtained by Tomi and Massimo.

Tomi thinks that the two measurements were equally inaccurate. Massimo thinks that one of the two measurements is more accurate than the other.

- Who do you agree with: Tomi or Massimo? Explain why.

Massimo decides to find the magnitude of the error as a percentage of the measured length.





- 3 Write down the measuring error in the length of one yard as a fraction of the exact length of the yard. Give your answer as percentage.
- 4 Write down the measuring error in the length of one foot as a fraction of the exact length of one foot. Give your answer as percentage.
- 5 **Conceptual** In what ways is expressing measuring errors as a percentage of the measured length helpful?
- 6 **Conceptual** How can measurement errors be compared?

When the exact value of a quantity is known, the error of a measured (approximate) value can be found as a percentage of the exact value:

Percentage error formula

$$\text{Percentage error} = \left| \frac{V_A - V_E}{V_E} \right| \times 100\%, \text{ where } V_A \text{ is the approximate}$$

(or measurement) value and V_E is the exact value.

Example 3

The fraction $\frac{22}{7}$ is often used as an approximation of π .

- a Find the percentage error of this approximation.
- b Find the least accurate decimal approximation of π needed to approximate π within 0.001% of the true value.

a Percentage error = $\left| \frac{\frac{22}{7} - \pi}{\pi} \right| \times 100\%$
 $= 0.0402\% \text{ (3 s.f.)}$

b $\left| \frac{V_A - \pi}{\pi} \right| \leq 0.00001$
 $-0.00001 \leq \frac{V_A - \pi}{\pi} \leq 0.00001$
 $\pi - 0.00001\pi \leq V_A \leq \pi + 0.00001\pi$

$$3.14156 \leq V_A \leq 3.14162$$

So $V_A = 3.1416$ will approximate π to within 0.001%

Note the absolute value creates a positive value.

Translating 0.001% to a decimal

$$|x| \leq C \text{ means } -C \leq x \leq C$$

Keep values exact until the last step, then evaluate π with technology.

Write enough digits of the upper and lower bounds to see where they differ.

TOK

In many cases, our measurements and calculations include errors. Very often courts rely on interpretations of forensic data and invite experts to courtrooms to give their opinion. If the percentage error of a certain DNA testing or drug testing to be found as 0.0526, can we be certain the person in question is guilty? What would be considered as an acceptable error rate especially when the stakes are so high?

Exercise 1B

- Find the range of possible values for the following measurements, which were rounded to the indicated degrees of accuracy:
 - 24 mm (nearest mm)
 - 3.2 m (tenth of a metre)
 - 1.75 kg (0.01 kg)
 - 1400 g (3 s.f.)
- In 1856, Andrew Waugh announced Mount Everest as 8840 m high, after several years of calculations based on observations made by the Great Trigonometric Survey. More recent surveys confirmed the height at 8848 m.
 - Assuming the more recent survey is an exact value, calculate the percentage error made in the earlier survey.
 - If the more recent survey was accurate to the nearest metre, find the range of possible values for the exact height of Mount Everest.
- Two lab groups in a Physics class measure the times for a ball to fall 1 metre and record the times in the following tables.

Group 1	
Trial	Time (s)
1	0.45
2	0.53
3	0.47
4	0.55
5	0.43
6	0.67

Group 2	
Trial	Time (s)
1	0.48
2	0.56
3	0.34
4	0.49
5	0.30
6	0.45

- Find the average of all the measurements for each group.
Using the laws of Physics the true value for the time of the fall is 0.452 seconds given air resistance can be ignored.
 - Calculate the percentage error for each set of data.
 - Based on your calculations, comment on the uncertainty of the results of each group.
- With 72 million bicycles, correct to the nearest million, Japan is at the top of the list of countries with most bicycles per capita. On average, Japanese people travel about 2 km, correct to the nearest km, on their bicycles each day. Calculate the upper bound for total distance travelled by all the bicycles in Japan per year.
 - To determine if a business is making enough profit the following formula is used $P = \frac{S - C}{S}$ where P is relative profit, S is sales income and C is costs. If a company has \$340 000 worth of sales and \$230 000 as costs, each correct to 2 significant figures, calculate the maximum and minimum relative profit to an appropriate degree of accuracy.



- 6 The temperature today in Chicago is 50°F . Being used to Celsius, Tommaso wants to convert the $^{\circ}\text{F}$ to $^{\circ}\text{C}$ to know what to wear outside. But instead of using the standard conversion formula $^{\circ}\text{C} = \frac{5}{9} \times (^{\circ}\text{F} - 32)$ he uses his grandmother's rule that is easier, but gives an approximate value: "Subtract 32° from the value in $^{\circ}\text{F}$ and multiply the result by 0.5 ".
- Calculate the actual and an approximate temperature value in $^{\circ}\text{C}$ using the standard formula and Tommaso's grandmother's rule.
 - Calculate the percentage error of the approximate temperature value, in $^{\circ}\text{C}$.
- 7 A factory produces circular slabs for use in construction. They guarantee that all slabs produced will have an area within 0.2% of the "target" of 163 m^2 .
- Find the range of values for the radius that will ensure all slabs produced are within this range.
 - Determine how accurately the radius must be measured during production to ensure that it will fall within this range.

Exponents and standard form

Exponents can make representing numbers and performing calculations easier and more exact. In particular, when dealing with very large or very small numbers, such as in astronomy, macroeconomics, or chemistry, **standard form** (or **scientific notation**) can be more efficient.

International-mindedness

Where did numbers come from?

Recall that a number can be written in the **standard form**

$$a \times 10^k \text{ with coefficient } 1 \leq a < 10 \text{ and exponent } k \in \mathbb{Z}.$$

Investigation 3

- 1 The three countries with the largest populations in 2017 were:

India: 1.34 billion

USA: 3.24×10^8

China: 1 409 517 397

- Convert all numbers to standard form. Round to 3 s.f. as needed.
- Explain how you can easily order these numbers from smallest to largest when they are converted to standard form.

Gross Domestic Product (GDP) measures the total value of goods and services produced by a country and is one way to measure the wealth of countries. The GDP per capita (per person) of three countries for 2017 is given in the table below.

Country	GDP per capita (\$/person)
India	1983
USA	59 501
China	8643

To find the total GDP of each country, multiply GDP per capita by population. First you will investigate how to multiply numbers in standard form by hand.



Continued on next page

- 2 Complete the examples below using technology:

	x	y	xy
(a)	3×10^5	2×10^9	
(b)	8×10^1	1×10^4	
(c)	2×10^{-3}	4×10^{17}	
(d)	5×10^6	3×10^{12}	
(e)	4×10^5	9×10^{-7}	

- 3 How can you find the product $(b \times 10^m)(c \times 10^n)$ in the form $a \times 10^k$ where $1 \leq a < 10$ and k is an integer? Make sure your process is consistent with all five examples above.
- 4 How does your process for multiplication relate to the law of exponents, $x^p \times x^q = x^{p+q}$?
- 5 Now estimate and calculate the GDP of each country.
- Write each GDP per capita in standard form.
 - Estimate the GDP of each country, without use of technology. Round your numbers as needed.
 - Use your GDC to calculate the GDP of each country.
 - Compare your estimates with the calculations. Was the magnitude (power of 10) of your estimate correct?
- 6 **Conceptual** How does standard form help with calculations?

HINT

Recall that you can enter numbers on your GDC in standard form: 3.4×10^7 is usually entered as 3.4 E 7. Make sure that you translate this calculator notation back to proper standard form when transferring answers from technology.

TOK

What might be the ethical implications of rounding numbers?

Numbers in standard form can be multiplied or divided following rules for exponents:

$$(b \times 10^m)(c \times 10^n) = bc \times 10^{m+n} \text{ and } \frac{b \times 10^m}{c \times 10^n} = \frac{b}{c} \times 10^{m-n}$$

After performing the operation, ensure your answer is given in the form $a \times 10^p$, where $0 \leq a < 10$ by adjusting the exponent as needed.

Example 4

Light travels at a speed of 3×10^8 m/s. The Earth is approximately 150 million kilometres from the Sun. Estimate the time, in seconds, that light takes to travel from the Sun to the Earth.

Since distance = speed \times time,

$$\begin{aligned} \text{time} &= \frac{1.50 \times 10^{11} \text{ m}}{3 \times 10^8 \text{ m/s}} \\ &= 0.5 \times 10^3 \text{ s} \\ &= 500 \text{ s} \end{aligned}$$

Convert the distance to metres in standard form:

$$\begin{aligned} 150 \text{ million km} &= 1.50 \times 10^8 \text{ km} \\ &= 1.50 \times 10^{11} \text{ m} \end{aligned}$$

Divide coefficients and subtract exponents.

Note that technology could also be used to divide.



A **negative exponent** represents a reciprocal power:

$$x^{-n} = \frac{1}{x^n}$$

A **rational exponent** represents a power of a root:

$$x^{\frac{p}{q}} = \sqrt[q]{x^p} = \left(\sqrt[q]{x}\right)^p, p, q \in \mathbb{Z}$$

In particular, $x^{\frac{1}{2}} = \sqrt{x}$.

The following rules of exponents hold for $a > 0$ and $m, n \in \mathbb{Q}$

$$a^m \times a^n = a^{m+n}$$

$$(a^m)^n = a^{mn}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

HINT

You should be able to evaluate exponents and roots by hand or with technology, such as:

$$\sqrt[3]{50} = 50^{\frac{1}{3}} \approx 3.68$$

$$2^{-4} = \frac{1}{2^4} = \frac{1}{16}$$

While we will frequently use technology to approximate values for rational exponents in this course, exact values allow us to calculate more precisely and to use rules of exponents.

Example 5



- a** Find the surface area of a cube with volume 50 cm^3
- exactly
 - approximately.
- b**
- Find a general formula for the surface area of a cube with volume $V \text{ cm}^3$.
 - Hence, determine the volume needed for a surface area of 1000 cm^2 .

a $V = 50 = s^3$, so $s = 50^{\frac{1}{3}}$.

$$SA = 6s^2 = 6\left(50^{\frac{1}{3}}\right)^2$$

i $= 6 \times 50^{\frac{2}{3}} \text{ cm}^2$

ii 81.4 cm^2

b **i** $SA = 6V^{\frac{2}{3}}$

ii $1000 = 6V^{\frac{2}{3}}$

$$\left(V^{\frac{2}{3}}\right)^{\frac{3}{2}} = \left(\frac{1000}{6}\right)^{\frac{3}{2}}$$

$$V = 2150 \text{ cm}^3 (3 \text{ s.f.})$$

Solve for the side length (s).

Substitute into the surface area formula.

Apply the rule $(a^m)^n = a^{mn}$.

Evaluate with technology.

Generalizing 50 to V .

Isolate V by dividing by six and then raising both sides to the reciprocal power. The exponent

on V becomes $\frac{2}{3} \times \frac{3}{2} = 1$

Example 6

A stuffed animal company finds that each store can sell stuffed bears for a price of $p = \frac{240}{\sqrt{x}}$, where x represents the population of the city in which the store operates, in thousands. Research also shows that the weekly quantity q of stuffed bears that will be sold can be found with the formula $q = 0.9x^{\frac{3}{4}}$. The total weekly revenue of a store is the product of its price and quantity sold.

- Determine an expression for the price in the form ax^m , where $a \in \mathbb{R}$ and $m \in \mathbb{Q}$.
- Determine store revenue in the form ax^m , where $a \in \mathbb{R}$ and $m \in \mathbb{Q}$.
- If the company wants to open a store that will make at least \$1500 per week, determine the smallest population of city they should consider.

$$\mathbf{a} \quad p = \frac{240}{\sqrt{x}} = 240x^{-\frac{1}{2}}$$

$$\mathbf{b} \quad r = 240x^{-\frac{1}{2}} \times 0.9x^{\frac{3}{4}} = 216x^{\frac{1}{4}}$$

$$\mathbf{c} \quad 1500 = 216x^{\frac{1}{4}}$$

$$x = \left(\frac{1500}{216}\right)^4 = 2330 \text{ (3 s.f.)}$$

The city should have a population of at least 2 330 000.

Multiply price by quantity. Multiply coefficients and add the exponents.

Isolate x .

“Undo” the $\frac{1}{4}$ exponent by raising both sides to the 4th power.

Interpret the solution; x is measured in thousands.

Exercise 1C



- For each operation, **i** estimate a value for the answer without technology, **ii** find the exact value using technology. Express all answers in the form $a \times 10^k$ where $1 \leq a < 10$ and $k \in \mathbb{Z}$.
 - $(1.08 \times 10^{-3})(9.2 \times 10^7)$
 - $\frac{7 \times 10^4}{7.24 \times 10^{-6}}$
- Calculate each expression using technology to 3 s.f. Express all answers in the form $a \times 10^n$ where $1 \leq a < 10$ and $k \in \mathbb{Z}$.
 - $(2.35 \times 10^{-6})(4 \times 10^1)$
 - $\frac{7.1 \times 10^6}{8.5 \times 10^2}$
 - $\frac{4}{3}\pi (5 \times 10^{-7})^3$
 - $\frac{50}{(8.8 \times 10^{-5})^{-2}}$
- Simplify each expression and write your solution
 - without negative exponents
 - in the form ab^c , where a , b , and c are values or variables.
 - $\frac{15x^{\frac{1}{2}}}{x}$
 - $7^{-\frac{1}{3}} \times 7^{\frac{4}{3}}$
 - $\frac{5 \times 2^{-3t}}{40}$
 - $\left(\frac{5}{3^x}\right)^2$
- There are initially 120 bacteria in a Petri dish and the population doubles approximately every hour, which can be represented by the formula $B = 120 \times 2^t$ where B is the number of bacteria and t is the time in hours since the bacteria began growing.
 - Find B when $t = 1$, $t = \frac{3}{2}$, and $t = 2$.

HINT

Remember to put brackets around numbers in standard form when you perform operations on them.



- b** Comment on what your answers tell you about the growth of the bacteria. What is the meaning of the value obtained when $t = \frac{3}{2}$.
- 5** The half-life of iodine-131 is approximately 8 days, which means the mass of a sample of iodine decays by half every 8 days. The amount remaining can be calculated using the formula $I = 1600 \times 2^{-\frac{t}{8}}$, where t is the time in days since the beginning of the sample.
- a** Write down this formula without negative exponents.
- b** Find the amount of material remaining after 4 days as an exact and approximate value.
- 6** The image of a speck of dust in an electron microscope is 1.2×10^2 mm wide. The image is 5×10^2 times larger than the actual size. Determine the width of the actual speck of dust.
- 7** The Earth's mass is 5.97×10^{24} kg and Mercury's mass is 3.29×10^{23} kg. How many times more massive is the Earth than Mercury?
- 8** The Earth's surface area is approximately 5.1×10^8 km² (2 s.f.) and its population is 7.6×10^9 (2 s.f.). **Population density** is the number of people per square kilometre. Determine the population density of the Earth assuming all the surface area is habitable by humans.
- About 30% of the Earth's surface is land, including Antarctica. Determine the population density of the earth assuming that all the land is habitable, but not the oceans.

1.2 Angles and triangles

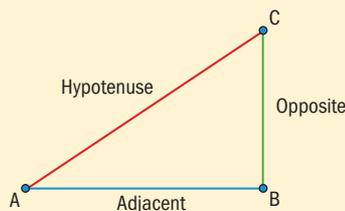
Trigonometric ratios in a right triangle

The ratios of the sides of a right-angled triangle are called **trigonometric ratios**. The three most common trigonometric ratios are the **sine (sin)**, **cosine (cos)**, and **tangent (tan)**. These are defined for acute angle A in the right-angled triangle below:

$$\sin(\hat{A}) = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\cos(\hat{A}) = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\tan(\hat{A}) = \frac{\text{Opposite}}{\text{Adjacent}}$$



In the definitions above, “opposite” refers to the length of the side opposite angle \hat{A} , “adjacent” refers to the length of the side adjacent to angle \hat{A} , and “hypotenuse” refers to the length of the hypotenuse (the side opposite the right angle).

HINT

Some people use the mnemonic **SOH-CAH-TOA**, pronounced “soh-kuh-toh-uh”, to help them remember the definitions of sine, cosine, and tangent.

International-mindedness

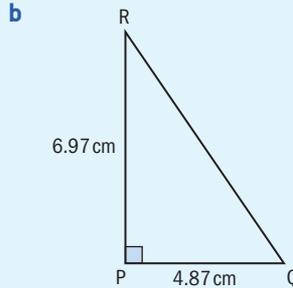
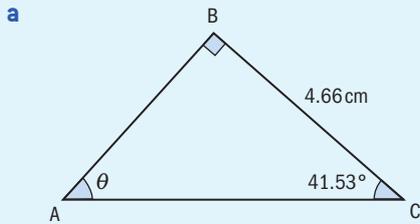
Diagrams of Pythagoras' theorem occur in early Chinese and Indian manuscripts. The earliest references to trigonometry are in Indian mathematics.

Reflect What are trigonometric ratios?

Example 7



For each triangle, solve for the unknown angles and sides.



a In $\triangle ABC$:

$$\theta = 90^\circ - 41.53^\circ = 48.47^\circ$$

$$\cos(41.53^\circ) = \frac{BC}{AC}$$

$$AC = \frac{4.66}{\cos(41.53^\circ)} \Rightarrow AC = 6.22 \text{ (3 s.f.)}$$

$$AB = \sqrt{AC^2 - BC^2} = 4.13 \text{ cm (3 s.f.)}$$

[BC] is adjacent to angle \hat{C} , use the cosine ratio to find hypotenuse AC. Ensure technology is set to degree mode. Round answers to 3 s.f.

Use Pythagoras' theorem to find AB.

Use the exact value of AC stored in your calculator (not the rounded value) when performing your calculations.

Use Pythagoras' theorem to find RQ.

b In $\triangle PQR$:

$$RQ = \sqrt{PR^2 + PQ^2} = 8.50 \text{ (3 s.f.)}$$

$$\tan(\hat{PQR}) = \frac{PR}{PQ} = \frac{6.97}{4.87}$$

$$\hat{PQR} = \tan^{-1}\left(\frac{6.97}{4.87}\right) = 55.1^\circ \text{ (3 s.f.)}$$

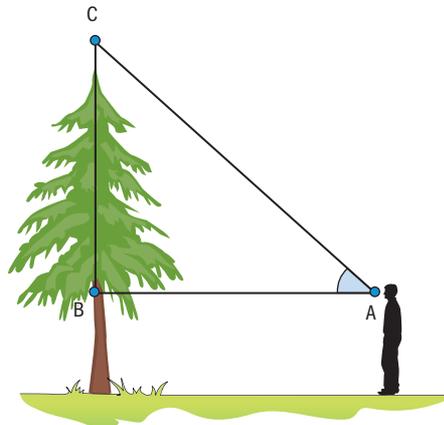
$$\hat{PRQ} = 90 - 55.1 = 34.9^\circ$$

When solving for an unknown angle, determine a trig ratio with two known sides (use exact lengths when possible). PR is opposite side to \hat{PQR} and PQ is adjacent, so choose tangent.

Then use an **inverse trig function**, in this case $\tan^{-1}(x)$, to find the angle.

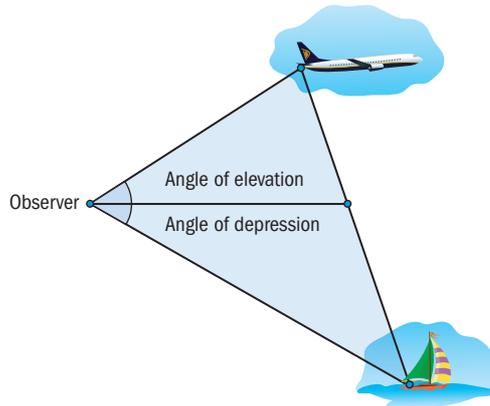
Angles of elevation and depression

Suppose an observer is standing in front of a tree, with their eyes at point A as shown in the diagram. Angle \hat{BAC} is formed when the observer looks up at the top of the tree. This angle is called an **angle of elevation** above the horizontal line at eye level (AB).





Similarly, when the object of sight falls below the horizontal at the eye level, an **angle of depression** is formed, as shown in the diagram.



Example 8

Emma stands 15 m away from a tree. She measures the angle of elevation to the top of the tree as 40° and her height to eye level as 1.42 m.

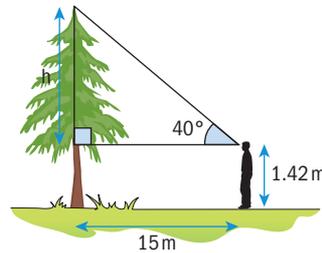
- Find the height of the tree.
- Frank, whose height is 1.8 m to eye level, is standing on the other side of the tree. His kite is stuck at the very top of the tree. He knows the length of the kite string is 16 m. What is the angle of elevation as Frank looks up at his kite?

$$\begin{aligned} \text{a } \tan 40^\circ &= \frac{h}{15} \\ h &= 15 \tan 40^\circ \end{aligned}$$

$$\begin{aligned} \text{Height of tree} &= 15 \tan 40^\circ + 1.42 \\ &= 14.0 \text{ m (3 s.f.)} \end{aligned}$$

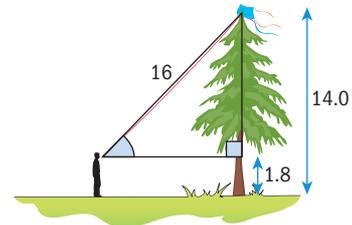
$$\text{b } \sin \theta = \frac{15 \tan 40^\circ + 1.42 - 1.8}{16} = \frac{12.206\dots}{16}$$

$$\theta = \sin^{-1}\left(\frac{12.206}{16}\right) = 49.7^\circ \text{ (3 s.f.)}$$



Add Emma's height to eye level.

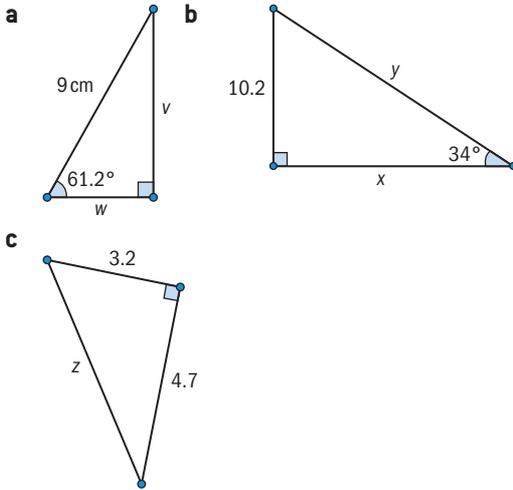
Subtract Frank's height from the height of the tree to find the opposite side length. Use the sine ratio as opposite and hypotenuse lengths are known.



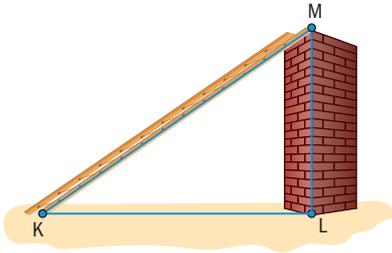
Use inverse sine (\sin^{-1}) to find the angle.

Exercise 1D

- 1 Determine all unknown sides and angles for each of the right triangles below:



- 2 A ladder [KM] is 8.5 m long. It currently leans against a vertical wall so that $\hat{LKM} = 30^\circ$.



- Find the distance KL.
 - Find how far up the wall the ladder reaches.
 - The instructions for use of the ladder state that the angle it makes with the ground should not exceed 55° . Find the maximum height that the ladder can reach up the wall.
- 3 A hiker, whose eye is 1.6 m above ground level, stands 50 m from the base of a vertical cliff. The angle between the line connecting her eye and the top of the cliff and a horizontal line is 58° .
- Draw a diagram representing the situation.
 - Find the height of the cliff.

- 4 The angle of depression from the top of a vertical cliff to a boat in the sea is 17° . The boat is 450 m from the shore.

- Draw a diagram.
- Find the height of the cliff. Give your answer rounded to the nearest metre.

- 5 Your family wants to buy an awning for a window that will be long enough to keep the sun out when it is at its highest point in the sky. The awning is attached to the wall at the top of the window and extends horizontally. The height of the window is 2.80 m. The angle of elevation of the sun at this point is 70° . Find how long the awning should be. Write your answer correct to 2 d.p.

- 6 Scientists measure the depths of lunar craters by measuring the shadow length cast by the edge of the crater on photographs. The length of the shadow cast by the edge of the Moltke crater is about 606 m, given to the nearest metre. The sun's angle of elevation "at the time the photograph was taken" is 65° . Find the depth of the crater. State your answer rounded to the nearest metre.

- 7 Maatsuyker Island Lighthouse is the last Australian lighthouse still being officially operated by lightkeepers. The lighthouse is 15 m high from its base to the balcony, and located 140 m above sea level.

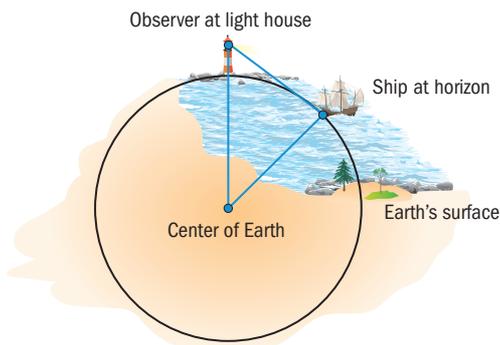


The caretaker is standing at the balcony and notices a ship at the horizon.

Find the straight line distance from the lighthouse balcony to the ship.



You may find it useful to draw a diagram like this



- 8 Hans is constructing an accessibility ramp for a library that should reach a height of 27 cm with an angle no greater than 13° .
- Find the shortest possible length of ramp to achieve this.
 - Hans cuts a length of wood to make the ramp. He cuts it to the length calculated in part **a** but can only cut it with an accuracy to the nearest centimetre. If the actual height required by the ramp is 27.43 cm, find the maximum possible percentage error between the desired 13° and the actual angle of the ramp.

Non-right triangles and the sine rule

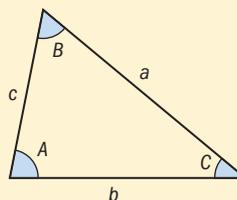
The trigonometric ratios we have used so far require a right triangle. If we have a non-right triangle, can we still find missing sides and angles?

Investigation 4

Part 1

Draw a scalene **obtuse** triangle $\triangle ABC$ (without a right angle) using dynamic geometry software, and label the vertices.

- Measure all the angles and side lengths of the triangle.
- Find the following **ratios** correct to 3 significant figures.



$$\frac{\sin \hat{A}}{a} = \frac{\sin \hat{B}}{b} = \frac{\sin \hat{C}}{c} =$$

What do you notice?

Part 2

Draw a scalene **acute** triangle $\triangle DEF$ (without a right angle) and label the vertices. Measure all the angles and side lengths

Find the following **ratios** correct to 3 significant figures

$$\frac{DE}{\sin \hat{F}} = \frac{EF}{\sin \hat{D}} = \frac{DF}{\sin \hat{E}} =$$

What do you notice?

Part 3

Repeat parts 1 and 2 for a right angled triangle. What do you notice?

Part 4

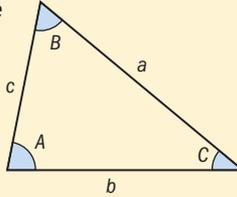
Are there any other types of triangle you could draw? Repeat Parts 1 and 2 for any other types of triangle you draw.

Conceptual What can you say about the ratio of the sine of an angle to the length of the side opposite the angle, in any triangle?

When is it most useful to use the sine rule with the angles “on top”? With the side lengths “on top”?

Sine Rule: In any non-right triangle, the ratio of each side to its opposite angle is the same for all three sides.

$$\frac{\sin \hat{A}}{a} = \frac{\sin \hat{B}}{b} = \frac{\sin \hat{C}}{c} \quad \text{or} \quad \frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}}$$



When solving a problem you will use just two of the three sides to set up an equation. When you are solving for a side length, it is easier to use the version with side lengths in the numerator, and similarly for angles.

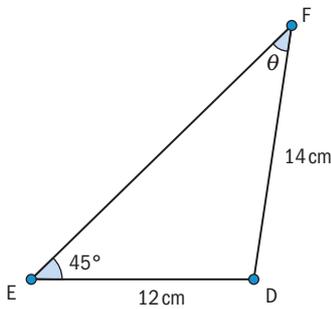
Example 9



In a triangle $\triangle DEF$, $DE = 12$ cm, $EF = 14$ cm and $\hat{D}EF = 45^\circ$.

Draw a labelled diagram and find the size of the angle $\hat{E}FD$ to the nearest degree.

Let $\hat{E}FD = \theta$



$$\frac{\sin 45^\circ}{14} = \frac{\sin \theta}{12}$$

$$\sin \theta = \frac{12 \sin 45^\circ}{14}$$

$$\theta = \sin^{-1}\left(\frac{12 \sin 45^\circ}{14}\right)$$

$$\theta = 37.30742828^\circ$$

$$\theta \approx 37^\circ \text{ (to the nearest degree)}$$

Use the Sine Rule with angles on top to solve for a missing angle more easily.

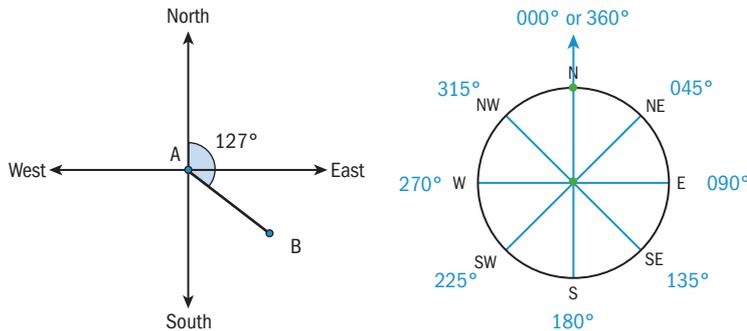
Solve this equation for θ using inverse sine.

Keep answers exact until the last step to avoid rounding errors.



Bearings

A bearing is an angle measured clockwise from North. The diagram on the left shows a bearing of 127° and the diagram on the right shows the bearings of the major compass directions.



International-mindedness

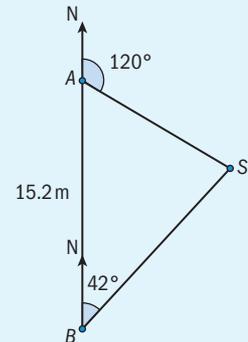
The word “sine” started out as a totally different word and passed through Indian, Arabic and Latin before becoming the word that we use today. Research on the Internet what the original word was.

Example 10

A ship S is located on a bearing of 120° from port A , and 042° from port B . The port A is directly North of the port B .

The distance between ports A and B is 15.2 miles.

Find the distance from the ship to each port.



$$\hat{A} = 180^\circ - 120^\circ = 60^\circ$$

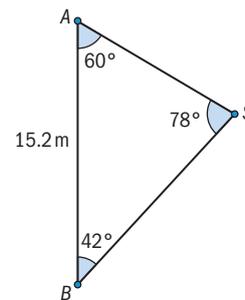
$$\hat{S} = 180^\circ - (42^\circ + 60^\circ) = 78^\circ$$

$$\frac{AS}{\sin 42^\circ} = \frac{BS}{\sin 60^\circ} = \frac{15.2}{\sin 78^\circ}$$

$$AS = \sin 42^\circ \times \frac{15.2}{\sin 78^\circ} = 10.4 \text{ miles (3 s.f.)}$$

$$BS = \sin 60^\circ \times \frac{15.2}{\sin 78^\circ} = 13.5 \text{ miles (3 s.f.)}$$

Use angle rules to find \hat{A} and \hat{S} .



Use the sine rule to find the lengths.

Triangulation

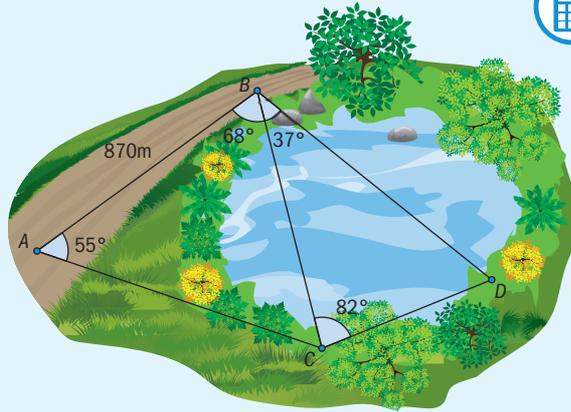
Surveyors extensively use **triangulation** to indirectly calculate large distances. By measuring the distance between two landmarks and the angles between those landmarks and a third point, the surveyor can calculate the other two distances in the triangle formed by those points. This process can then be repeated to form a chain of triangles and is illustrated in the following example.

TOK

What does it mean to say that mathematics is an axiomatic system?

Example 11

The diagram shows a lake with three docks at points B, C and D. The distance AB along a highway is known to be 870 m. Surveyors measure the angles as given in the diagram.



- Use triangulation to find the distances BC and BD.
- Nils, who rows at a speed of 1.5 m/s, starts from dock B. Calculate how much longer will it take him to cross the lake if he rows to the further of the two docks.

$$\text{a } \hat{C} = 180 - 55 - 68 = 57^\circ$$

$$\frac{BC}{\sin 55^\circ} = \frac{870}{\sin 57^\circ}$$

$$BC = \frac{870 \times \sin 55^\circ}{\sin 57^\circ} = 950 \text{ m (3 s.f.)}$$

$$\frac{BC}{\sin 61^\circ} = \frac{BD}{\sin 82^\circ}$$

$$BD = \frac{BC \times \sin 82^\circ}{\sin 61^\circ} = 962 \text{ m (3 s.f.)}$$

$$\text{b } \frac{BD - BC}{1.5} = \frac{112 \text{ m}}{1.5} = 74.9 \text{ s}$$

Find the angle opposite the known side,

$$\hat{C} = 180 - \hat{A} - \hat{ABC}$$

Substitute into the Sine Rule to find BC.

Repeat the process with angles \hat{BCD} and \hat{D} . Remember to use the "exact" value of BC from your calculator.

$$\text{Using time} = \frac{\text{distance}}{\text{speed}}$$

Investigation 5

- Use the Sine Rule to find \hat{D} in the diagram on the right:
- Explain why the solution you've obtained is not consistent with the diagram.

The issue you have just encountered is known as the **Ambiguous Case** of the Sine Rule. When **two sides and the non-included angle** are known, and the **unknown angle is opposite the longer of the two sides**, then two triangles are possible:

