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MATHEMATICS:
ANALYSIS AND APPROACHES

STANDARD LEVEL
COURSE COMPANION

ENHANCED ONLINE

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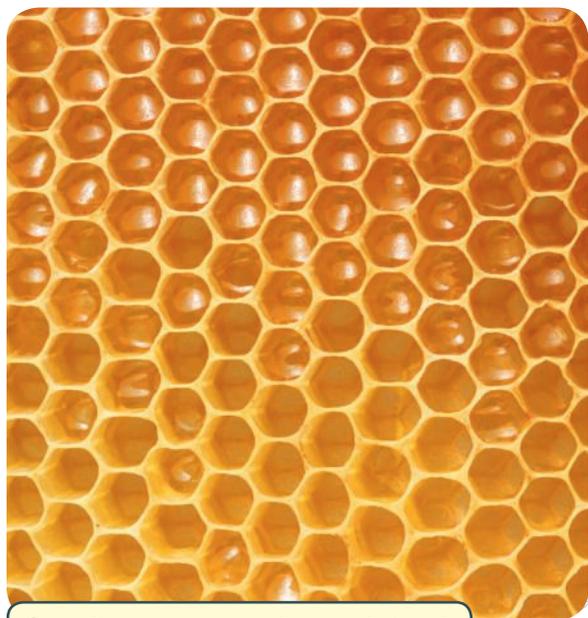
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Contents

1 From patterns to generalizations: sequences and series

You do not have to look far and wide to find visual patterns—they are everywhere!

Concepts
- Patterns
- Generalization

Microconcepts
- Arithmetic and geometric sequences
- Arithmetic and geometric series
- Common difference
- Sigma notation
- Common ratio
- Sum of sequences
- Binomial theorem
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- Sum to infinity

Can these patterns be explained mathematically?

Can patterns be useful in real-life situations?

What information would you require in order to choose the best loan offer? What other scenarios could this be applied to?

If you take out a loan to buy a car how can you determine the actual amount it will cost?

The diagrams shown here are the first four iterations of a fractal called the Koch snowflake.

What do you notice about:

- how each pattern is created from the previous one?
- the perimeter as you move from the first iteration through the fourth iteration? How is it changing?
- the area enclosed as you move from the first iteration to the fourth iteration? How is it changing?

What changes would you expect in the fifth iteration?

How would you measure the perimeter at the fifth iteration if the original triangle had sides of 1 m in length?

If this process continues forever, how can an infinite perimeter enclose a finite area?
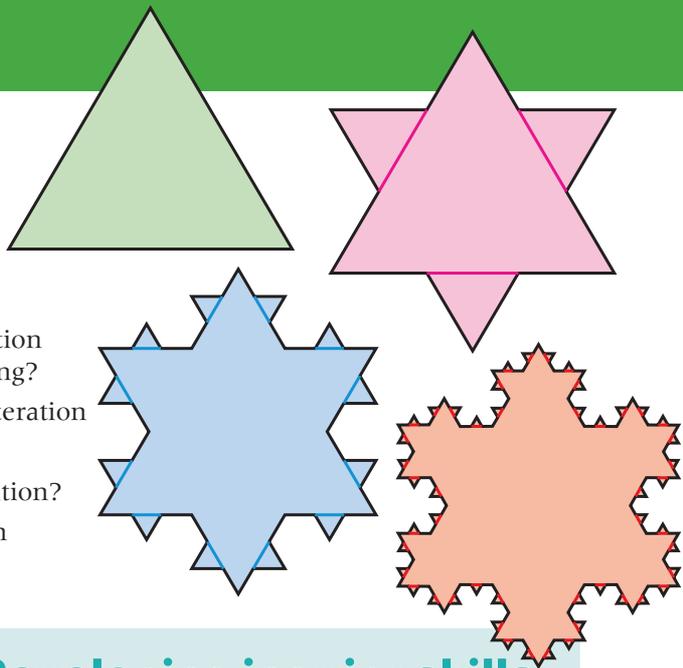
Developing inquiry skills

Does mathematics always reflect reality? Are fractals such as the Koch snowflake invented or discovered?

Think about the questions in this opening problem and answer any you can. As you work through the chapter, you will gain mathematical knowledge and skills that will help you to answer them all.

Before you start

You should know how to:

1 Solve linear equations:

eg $2(x - 2) - 3(3x + 7) = 4$

$$2x - 4 - 9x - 21 = 4$$
$$-7x - 25 = 4$$
$$-7x = 29$$
$$x = \frac{-29}{7}$$

2 Perform operations $(+ - \times \div)$ with fractions and simplify using order of operations:

eg $\frac{1}{2} + \frac{3}{4} \times \frac{2}{5}$

$$= \frac{1}{2} + \frac{6}{20} = \frac{1}{2} + \frac{3}{10} = \frac{5}{10} + \frac{3}{10} = \frac{8}{10} = \frac{4}{5}$$

3 Substitute into a formula and simplify using order of operations:

eg $P = -2(x + 4)^2 + 3x, x = -1$

$$P = -2(-1 + 4)^2 + 3(-1)$$
$$P = -2(3)^2 + 3(-1)$$
$$P = -2(9) + 3(-1)$$
$$P = -18 - 3$$
$$P = -21$$

Skills check

1 Solve each equation:

  a $2x + 10 = -4x - 8$

  b $3a - 2(2a + 5) = -12$

  c $(x + 2)(x - 1) = (x + 5)(x - 2)$

2 Simplify:

  a $\frac{3}{4} + \frac{1}{2} - \frac{5}{6}$ **b** $\frac{1}{2} \times \frac{3}{4} + \frac{5}{6} \times \frac{7}{8}$ **c** $\frac{-3}{4} \div \frac{5}{8}$

3 Substitute the given value(s) of the variable(s) and simplify the expression using order of operations:

  a $A = 2(x - 3)^2, x = -5$

  b $S = 2^x + (x - 1)(x + 3), x = 2$

  c $P = 3d - 2(n + 4)^2, d = -3, n = 2$

1.1 Number patterns and sigma notation

Sequences

Investigation 1

Luis wants to join a gym. He finds one that charges a $100 membership fee to join and $5 every time he uses the facilities. What formula will help Luis to calculate how much he will pay in a given number of visits? If he goes to the gym 12 times in the first month, how much will he pay that month?

Luis's friend Elijah decides to join a different gym. His has no membership fee but charges $15 per use. How many times does each of them need to use their gym before they have paid the same total amount?

1 Complete the table below of Luis's and Elijah's gym fees:

Number of visits to the gym	Total fees paid by Luis	Total fees paid by Elijah
0	$100	$0
1	$105	$15
2	$110	$30
3		
4		
5		
6		
7		
8		

2 What patterns do you see emerging?

3 Which pattern is increasing faster?

4 When will the fees paid by Luis and Elijah be the same?

5 Are there any limitations to these patterns? Will they eventually reach a maximum value?

The fees paid in the investigation above create a sequence of numbers.

> A **sequence** is a list of numbers that is written in a defined order, ascending or descending, following a specific rule.

Look at the following sequences and see if you can determine the next few terms in the pattern:

i 2, 4, 6, 8, …

ii 14, 11, 8, 5, …

iii 1, 4, 9, 16, …

iv 1, 3, 9, 27, …

A sequence can be either finite or infinite.

> A **finite sequence** has a fixed number of terms.
>
> An **infinite sequence** has an infinite number of terms.

For example, 7, 5, 3, 1, −1, −3 is finite because it ends after the sixth term.

For example, 7, 5, 3, −1, … is infinite because the three dots at the end (called an ellipsis) indicate that the sequence is never ending or continues indefinitely.

A sequence can also be described in words. For example, the positive multiples of 3 less than 40 is a finite sequence because the terms are 3, 6, 9, … 39.

A term in a sequence is named using the notation u_n, where n is the position of the term in the sequence. The first term is called u_1, the second term is u_2, etc.

> A formula or expression that mathematically describes the pattern of the sequence can be found for the general term, u_n.

Let's return to the opening investigation and determine an expression for the general term for each sequence.

Investigation 1 continued

1   Write down the first few terms for Luis.

2   Describe the pattern.

3   Justify the general expression (u_n) for Luis at the bottom of the table below.

4   Write down the first few terms for Elijah.

5   Describe the pattern.

6   Complete the table below:

Term number	Total fees paid by Luis	Pattern for Luis	Total fees paid by Elijah	Pattern for Elijah
u_1	$100	$100 + (5 \times 0)$	$0	15×0
u_2	$105	$100 + (5 \times 1)$	$15	
u_3	$110	$100 + (5 \times 2)$	$30	
u_4	$115	$100 + (5 \times 4)$	$45	
u_5	$120	$100 + (5 \times 0)$	$60	
u_n	—	$100 + 5(n-1)$	—	

7   The general term for Luis can be simplified from $u_n = 100 + 5(n-1)$ to $u_n = 5n + 95$. Justify this simplification.

8 What is the simplified form of Elijah's general term?

9 How did the pattern help you to write the general term for Elijah's sequence?

A sequence is called **arithmetic** when the same value is added to each term to get the next term.

For example,

$$\overset{+5}{\frown}\ \overset{+5}{\frown}\ \overset{+5}{\frown}\ \overset{+5}{\frown}$$
100, 105, 110, 115, 120, …

Another example of an arithmetic sequence is 1, 4, 7, 10, … because 3 is added to each term. The value added could also be negative, such as −5. For example, 20, 15, 10, 5, …

Investigation 2

Let's look at a new type of sequence.

1 Complete this table of values for the next few terms of each sequence:

Sequence 1	Sequence 2	Sequence 3
5	2	36
10	−4	12
20	8	4
40	−16	$\frac{4}{3}$

2 Describe the pattern. For any of these sequences, how do you calculate a term using the previous term?

Look at the pattern for the first sequence as shown in the table below:

Term number	Sequence 1	Pattern	Sequence 2	Pattern	Sequence 3	Pattern
u_1	5	$5 \times 1 = 5 \times 2^0$	2		36	
u_2	10	$5 \times 2 = 5 \times 2^1$	−4		12	
u_3	20	$5 \times 4 = 5 \times 2^2$	8		4	
u_4	40	$5 \times 8 = 5 \times 2^3$	−16		$\frac{4}{3}$	
u_5	80	$5 \times 16 = 5 \times 2^4$	32		$\frac{4}{9}$	
u_n	—	$5 \times 2^{n-1}$	—		—	

3 Describe the relationship between the term number and the power of 2 for that term.

4 Justify the expression for the general term for the first pattern.

5 Complete the table for the other two patterns.

6 **Factual** How do you find the general term of **this type** of sequence?

7 **Conceptual** How does looking at the pattern help you find the general term of any sequence?

> A sequence is called **geometric** when each term is multiplied by the same value to get the next term.

For example,
$$\underset{\times 2 \;\; \times 2 \;\; \times 2}{5,\ 10,\ 20,\ 40,\ \dots} \quad \text{or} \quad \underset{\times \frac{1}{2} \;\; \times \frac{1}{2} \;\; \times \frac{1}{2}}{200,\ 100,\ 50,\ 25,\ \dots}$$

Example 1

Find an expression for the general term for each of the following sequences and state whether they are arithmetic, geometric or neither.

a 3, 6, 9, 12, … **b** 3, −12, 48, −192, … **c** 2, 10, 50, 250, … **d** $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \dots$

a The sequence can be written as $3 \times 1, 3 \times 2, 3 \times 3, \dots$	This sequence is the positive multiples of 3.
The general term is $u_n = 3n$.	This is an arithmetic sequence because 3 is being added to each term.
This is an arithmetic sequence.	
b The sequence can be written as $3 \times 1, 3 \times -4, 3 \times 16, 3 \times -64$ which can be expressed as $3 \times (-4)^0, 3 \times (-4)^1, 3 \times (-4)^2,$ $3 \times (-4)^3, \dots$	This is a geometric sequence because each term is multiplied by −4.
The general term is $u_n = 3(-4)^{n-1}$	
This is a geometric sequence.	
c The sequence can be written as $2 \times 1, 2 \times 5, 2 \times 25, 2 \times 125, \dots$ or $2 \times 5^0, 2 \times 5^1, 2 \times 5^2, 2 \times 5^3, \dots$	This is a geometric sequence because each term is multiplied by 5.
The general term is $u_n = 2 \times 5^{n-1}$	
This is a geometric sequence.	

d The sequence can be written as $\dfrac{1}{3}$, $\dfrac{2}{2+3}$, $\dfrac{3}{4+3}$, $\dfrac{4}{6+3}$, ... or

$$\dfrac{1}{3+(2\times 0)}, \dfrac{2}{3+(2\times 1)}, \dfrac{3}{3+(2\times 2)}, \dfrac{4}{3+(2\times 3)}, ...$$

The general term is

$$u_n = \dfrac{n}{3+2(n-1)}$$
$$= \dfrac{n}{3+2n-2}$$
$$= \dfrac{n}{2n+1}$$

The sequence is neither arithmetic nor geometric.

Consider the numerators and denominators as two separate sequences.	

The numerators are the positive integers.

The denominators, 3, 5, 7, 9, ... , are the odd positive integers with $u_1 = 3$ and 2 being added to each term.

Each sequence has a first term, a second term, a third term, etc, so n will always be a positive integer.

Exercise 1A

1 Write down the next three terms in each sequence:

a −8, −11, −14, −17, ... **b** 9, 16, 25, 36, ...

c 6, 12, 18, 24, ... **d** 1000, 500, 250, 125, ...

e $\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, ...$ **f** $\dfrac{1}{3}, \dfrac{2}{9}, \dfrac{3}{27}, \dfrac{4}{81}, ...$

2 For each of the following sequences, find an expression for the general term and state whether the sequence is arithmetic, geometric or neither:

a 10, 50, 250, 1250 **b** 41, 35, 29, 23, ...

c $\dfrac{1}{3}, -\dfrac{1}{9}, \dfrac{1}{27}, -\dfrac{1}{81}, ...$ **d** 1, 1, 2, 3, 5, 8, ...

e $\dfrac{1}{2}, \dfrac{3}{4}, \dfrac{5}{6}, \dfrac{7}{8}, ...$ **f** −12, −36, −108, −324, ...

3 For each of the following real-life situations, give the general term and state whether the sequence is arithmetic, geometric or neither.

a Ahmad deposits $100 in a savings account every month. After the first month, the balance is $100; after the second month, the balance is $200; and so on.

b Luciana is trying to lose weight. The first month, she loses 6 kg, and she continues to lose half as much each subsequent month.

c The temperature of the water in the swimming pool in your backyard is only 70°F. It is too cold, so you decide to turn up the temperature by 10% every hour until it is warm enough.

Consider the sequence −4, −3, −1, 3, ... Notice that it is neither arithmetic or geometric. This is a new type of sequence called recursive.

> A **recursive sequence** uses the previous term or terms to find the next term. The general term will include the notation u_{n-1}, which means "the previous term".

For example, given the general term $u_n = 2u_{n-1} + 5$ and $u_1 = -4$, the second term can be found by replacing u_{n-1} in the general term with u_1:

$$u_2 = 2u_1 + 5 = 2(-4) + 5 = -3$$

The third term can then be found by repeating the process with u_2:

$$u_3 = 2u_2 + 5 = 2(-3) + 5 = -1$$

Here is another example:

Given $u_n = (u_{n-1})^2 - 2$ and $u_1 = 5$, find the next three terms in the sequence.

$$u_2 = (u_1)^2 - 2 = 5^2 - 2 = 23$$
$$u_3 = (u_2)^2 - 2 = 23^2 - 2 = 527$$
$$u_4 = (u_3)^2 - 2 = 527^2 - 2 = 277\,727$$

Example 2

Find the first five terms for each of the following recursive sequences:

a $u_n = \dfrac{u_{n-1}}{3} + 3$ and $u_1 = 9$ 　　　　　**b** $u_n = 2u_{n-1} - 3$ and $u_1 = 1$

a $u_2 = \dfrac{u_1}{3} + 3 = \dfrac{9}{3} + 3 = 3 + 3 = 6$

To find the second term, replace u_{n-1} in the general term with the value of u_1 given in the question.

$u_3 = \dfrac{u_1}{3} + 3 = \dfrac{6}{3} + 3 = 2 + 3 = 5$

To find the third term, now replace u_{n-1} with the answer you found for u_2.

$u_4 = \dfrac{u_3}{3} + 3 = \dfrac{5}{3} + 3 = \dfrac{5}{3} + \dfrac{9}{3} = \dfrac{14}{3}$

Repeat this process until you have the first five terms.

$u_5 = \dfrac{u_4}{3} + 3 = \dfrac{\frac{14}{3}}{3} + 3 = \dfrac{14}{9} + \dfrac{27}{9} = \dfrac{41}{9}$

$9, 6, 5, \dfrac{14}{3}, \dfrac{41}{9}$

b $u_2 = 2u_1 - 3 = 2(1) - 3 = -1$
$u_3 = 2u_2 - 3 = 2(-1) - 3 = -5$
$u_4 = 2u_3 - 3 = 2(-5) - 3 = -13$
$u_5 = 2u_4 - 3 = 2(-13) - 3 = -29$

$1, -1, -5, -13, -29$

Example 3

For each of the recursive sequences below, find a recursive formula for the general term:

a 12, 8, 4, 0, …   **b** −0.32, 3.2, −32, 320, ...   **c** 500, 100, 20, 4, …

a To calculate the next term, add −4. $u_n = u_{n-1} - 4$, where $u_1 = 12$	First, consider whether the sequence is arithmetic or geometric. This can be determined by checking whether a value is being added or multiplied to get the consecutive terms. Since −4 is being added each time, the recursive general term is $u_n = u_{n-1} - 4$, where $u_1 = 12$. It is important to state the first term so that you know the first value to substitute into the general term.
b To calculate the next term, multiply by −10. $u_n = -10u_{n-1}$, where $u_1 = -0.32$	
c To calculate the next term, multiply by $\frac{1}{5}$. $u_n = \frac{1}{5}u_{n-1}$, where $u_1 = 500$	

Exercise 1B

1 Find the first five terms for each of the following recursive sequences.

a $u_n = -4u_{n-1}$, $u_1 = 1$

b $u_n = \dfrac{-2}{u_{n-1}}, u_1 = 3$

c $u_n = 2(u_{n-1})^2$, $u_1 = -1$

d $u_n = 3u_{n-1} + 5$, $u_1 = m$

2 For each of the recursive sequences below, find a recursive formula for the general term.

a −2, −4, −6, −8, …

b 1, 4, 64, 256, …

c 52, 5.2, 0.52, 0.052, …

d 14, 19, 24, 29, …

e 2, 3, 6, 18, 108, 1944, …

f 1, 2, 6, 24, …

Series

A **series** is created when the terms of a sequence are added together.

A sequence or series can be either finite or infinite.

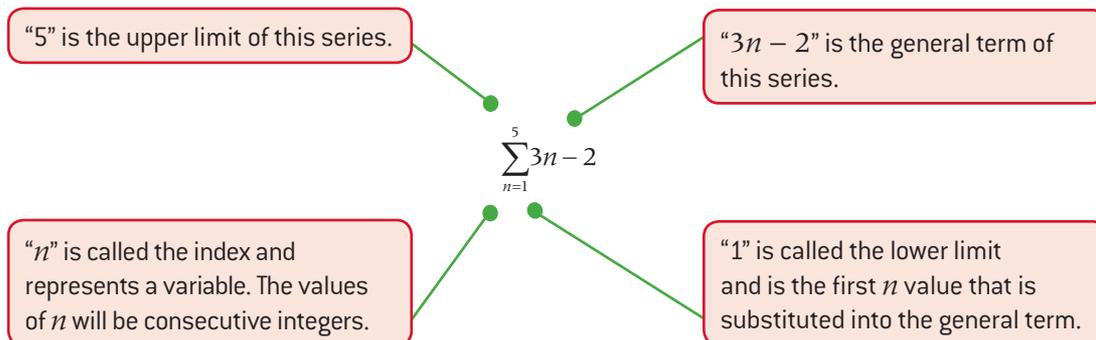A **finite series** has a fixed number of terms.

For example, $7 + 5 + 3 + 1 + -1 + -3$ is finite because it ends after the sixth term.

An **infinite series** continues indefinitely.

For example, the series $10 + 8 + 6 + 4 + $ … is infinite because the ellipsis indicates that the series continues indefinitely.

A series can be written in a form called **sigma notation**.

Sigma is the 18th letter of the Greek alphabet, and the capital letter, Σ, is used to represent a sum. Here is an example of a finite series written in sigma notation:

"5" is the upper limit of this series.

"$3n - 2$" is the general term of this series.

$$\sum_{n=1}^{5} 3n - 2$$

"n" is called the index and represents a variable. The values of n will be consecutive integers.

"1" is called the lower limit and is the first n value that is substituted into the general term.

Consecutive n values are substituted, until the "5" or upper limit is reached. It will be the last value substituted.

$$\sum_{n=1}^{5} 3n - 2 = \left(3(1) - 2\right) + \left(3(2) - 2\right) + \left(3(3) - 2\right) + \left(3(4) - 2\right) + \left(3(5) - 2\right)$$

$$\sum_{n=1}^{5} 3n - 2 = 1 + 4 + 7 + 10 + 13 = 35$$

TOK

Is mathematics a language?

For an infinite series, the upper limit is ∞. An example of an infinite geometric series is $\sum_{n=1}^{\infty} 3 \times 2^n$.

Example 4

For each of the following finite series in sigma notation, find the terms and calculate the sum:

a $\quad \sum_{n=1}^{4} (-1)^n n^2$ **b** $\quad \sum_{n=3}^{7} (-2)^n$ **c** $\quad \sum_{n=1}^{3} 2n - n^2$

a $\quad (-1)^1 (1)^2 + (-1)^2 (2)^2 + (-1)^3 (3)^2 + (-1)^4 (4)^2$ $\qquad = -1 + 4 - 9 + 16$ $\qquad = 10$	Substitute $n = 1$ to find the first term, $n = 2$ for the second term and so on. The last term is found when you substitute the upper limit of $n = 4$. Remember to add up all the terms once you have found them.
b $\quad (-2)^3 + (-2)^4 + (-2)^5 + (-2)^6 + (-2)^7$ $\qquad = (-8) + 16 + (-32) + 64 + (-128)$ $\qquad = -88$	Substitute $n = 3$ to find the first term, $n = 4$ for the second term and so.
c $\quad (2(1) - 1^2) + (2(2) - 2^2) + (2(3) - 3^2)$ $\qquad = 1 + 0 + (-3)$ $\qquad = -2$	

Example 5

Write each of the following series in sigma notation:

a $-3 + 5 + 13 + 21 + 29$ **b** $-3 + 6 - 12 + 24$ **c** $8 + 12 + 16 + 20 + \dots$

a   This finite arithmetic series can be rewritten as:

$-3 + (-3 + 8) + (-3 + 16) + (-3 + 24) + (-3 + 32)$

$\quad = (-3 + 0 \times 8) + (-3 + 1 \times 8) + (-3 + 2 \times 8)$
$\quad\quad + (-3 + 3 \times 8) + (-3 + 4 \times 8)$

The general term is $-3 + 8n$.

Therefore, the sigma notation is $\displaystyle\sum_{n=0}^{4} -3 + 8n$.

b   This finite geometric series can be rewritten as $-3 \times 1 + 3 \times 2 + -3 \times 4 + 3 \times 8$
$= -3 \times 2^0 + 3 \times 2^1 + -3 \times 2^2 + 3 \times 2^3$

The general term must include 3×2^n.

Since the first term is negative and the signs of following terms alternate, you need to multiply by $(-1)^{n+1}$.

If the first term was positive, or if you were starting with $n = 1$, you would multiply by $(-1)^n$ to make the signs alternate.

Therefore, the sigma notation is:

$$\sum_{n=0}^{3} 3(-1)^{n+1} 2^n$$

The answers to parts **a** and **b** are just one way to write each sequence using sigma notation. For example, another correct expression for part **a** would be

$$\sum_{n=1}^{5} -3 + 8(n-1)$$

c $\displaystyle\sum_{n=2}^{\infty} 4n$

or

$$\sum_{n=1}^{\infty} 4(n+1)$$

This infinite arithmetic series is the multiples of 4, starting with the second multiple.

To make the lower limit 1, you can also write the notation like this.

Note that because this is an infinite series, the upper limit is ∞.

Exercise 1C

1   For each of the following series in sigma notation, find the terms and calculate the sum:

a $\displaystyle\sum_{n=1}^{4} (-1)^n (n+1)$ **b** $\displaystyle\sum_{n=2}^{6} 4n - 3$

c $\displaystyle\sum_{n=1}^{3} n(n+1)$ **d** $\displaystyle\sum_{n=3}^{5} \frac{(-1)^{n+1}}{n-2}$

2   Write each of the following series in sigma notation:

a $4 + 16 + 64 + 256 + \dots$

b $\dfrac{3}{4} + \dfrac{4}{5} + \dfrac{5}{6}$ **c** $-1 + \dfrac{1}{2} - \dfrac{1}{3} + \dfrac{1}{4} - \dots + \dfrac{1}{100}$

**d** $-2 - 2 - 2 - 2 - 2 - 2 - 2 - 2$

e $5 + 10 + 17 + 26 + \dots$

f $49m^6 + 64m^7 + 81m^8 + 100m^9 + 121m^{10}$

Developing inquiry skills

Let's return to the chapter opening problem about the Koch snowflake.

 i How many sides does the initial triangle have?
 ii How many sides does the second iteration have? What about the third iteration?
 iii What kind of sequence do these numbers form?
 iv Write the general term for the number of sides in any iteration.
 v You can use your general term to find the number of sides for a given iteration by substituting the value for n. Find the number of sides the 12th iteration will have.
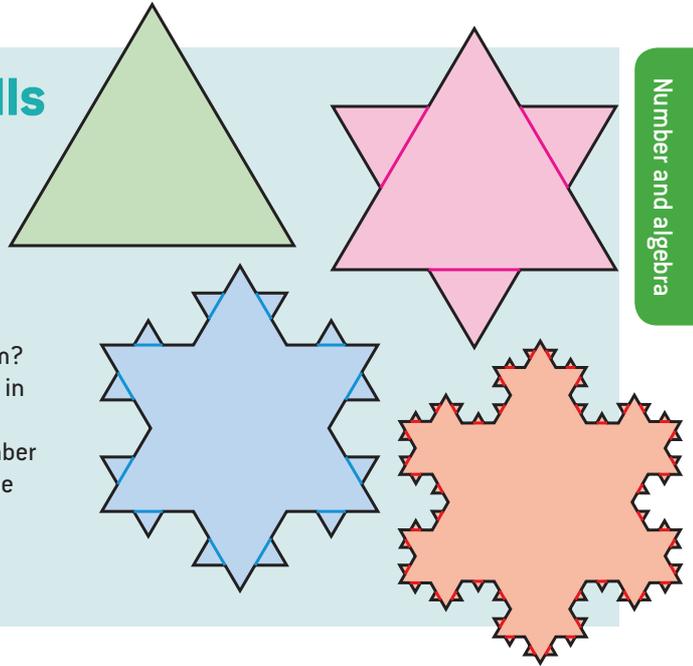
1.2 Arithmetic and geometric sequences

Investigation 3

A restaurant is discussing how many people they can fit at their tables. One table seats four people. If they push two tables together, they can seat six people. If three tables are joined, they can seat eight people.

1 Copy and complete the table as shown below:

Number of tables	1	2	3	4	5	6	7	8
Number of seats	4	6	8					

2 For every new table added, how many extra seats are created?

3 Write a general formula to calculate the number of seats for any number of joined tables.

4 You can use your general term to find the number of seats for a given number of tables by substituting the value for n in your general formula. How many seats will there be with twenty tables?

Let's revisit arithmetic sequences (progressions). From the last section and the investigation above, a sequence is arithmetic when each term after the first is found by adding a fixed non-zero number. This number is called the **common difference (d)**.

The common difference can be found by taking any term and subtracting the previous term: $d = u_n - u_{n-1}$.

When $d > 0$, the sequence is **increasing**.

When $d < 0$, the sequence is **decreasing**.

Let's derive the general formula for any arithmetic sequence:

$u_1, u_2, u_3, \ldots$

$= u_1, u_1 + d, u_1 + d + d, u_1 + d + d + d, \ldots$

$= u_1, u_1 + d, u_1 + 2d, u_1 + 3d, \ldots$

> Looking at the pattern between the term number and the coefficient of d, the formula for any term in the sequence is
> $$u_n = u_1 + (n-1)d$$

Example 6

For each arithmetic progression, write down the value of d and find the term indicated.

a $14, 11, 8, \ldots$ Find u_{12} **b** $2, 7, 12, \ldots$ Find u_9 **c** $-7, -5, -3, \ldots$ Find u_{21}

a $d = 8 - 11$ or $11 - 14 = -3$

$u_n = u1 + (n-1)d$

$u_{12} = 14 + (12-1)(-3)$

$u_{12} = 14 + (11)(-3)$

$u_{12} = 14 - 33$

$u_{12} = -19$

b $d = 12 - 7$ or $7 - 2 = 5$

$u_n = u_1 + (n-1)d$

$u_9 = 2 + (9-1)(5)$

$u_9 = 2 + (8)(5)$

$u_9 = 2 + 40$

$u_9 = 42$

To find d, take any term and subtract the previous term.

Substitute u_1, n and d into the general term formula and simplify.

c $d = -3 - (-5)$ or $-5 - (-7) = 2$
$u_n = u_1 + (n-1)d$
$u_{21} = -7 + (21-1)(2)$
$u_{21} = -7 + (20)(2)$
$u_{21} = -7 + 40$
$u_{21} = 33$

Exercise 1D

For each sequence below, use the general formula to find the term indicated.

1 $5, 13, 21, \ldots u_9$

2 $40, 32, 24, \ldots u_{11}$

3 $5.05, 5.37, 5.69, \ldots u_7$

4 $\dfrac{1}{2}, \dfrac{5}{6}, \dfrac{7}{6}, \ldots u_6$

5 $x+2, x+5, x+8, \ldots u_9$

6 $3a, 6a, 9a, \ldots u_{12}$

The general formula can be used to find more than just a given term. It can also be used to solve for u_1, n or d.

Example 7

Given an arithmetic sequence in which $u_1 = 14$ and $d = -3$, find the value of n such that $u_n = 2$

$u_n = u_1 + (n-1)d$ $2 = 14 + (n-1)(-3)$ $2 = 14 - 3n + 3$ $-3n = -15$ $n = 5$	Substitute u_1, u_n and d into the general term formula and solve for n. When solving for n, remember that your answer needs to be a natural number!

Example 8

Find the number of terms in the following finite arithmetic sequence.
$-8, -4, 0, \ldots 36$

$u_n = u_1 + (n-1)d$ $36 = -8 + (n-1)(4)$ $44 = 4n - 4$ $48 = 4n$ $n = 12$	Substitute u_1, u_n and d into the general term formula and solve for n.

Example 9

Two terms in an arithmetic sequence are $u_6 = 4$ and $u_{11} = 34$. Find u_{15}.

$u_6 + 5d = u_{11}$	The issue here is that you do not have u_1 or d. However, you do know that u_{11} is five terms higher in the sequence than u_6. Therefore, the difference between these terms is $5d$.
$4 + 5d = 34$ $5d = 30$ $d = 6$	Substitute values for the terms and rearrange to find d.
$u_{15} = u_{11} + 4d$ $u_{15} = 34 + 4(6)$ $u_{15} = 58$	Similarly, u_{15} is four terms higher in the sequence than u_{11}.

Exercise 1E

1 Given an arithmetic progression with $u_{21} = 65$ and $d = -2$, find the value of the first term.

2 Given that two terms of an arithmetic sequence are $u_5 = -3.7$ and $u_{15} = -52.3$, find the value of the 19th term.

3 Given an arithmetic sequence in which $u_1 = 11$ and $d = -3$, find the term that has a value of 2.

4 There exists an arithmetic sequence with $t_3 = 4$ and $t_6 = 184$. Find the 14th term.

5 Find the number of terms in the finite arithmetic sequence 6, −1, −8, … , −36.

6 A movie theatre with 12 rows of seating has 30 seats in the first row. Each row behind it has two additional seats. How many seats are in the last row?

7 The Winter Olympics are held every four years and were held in Vancouver, Canada in 2010. When will the Winter Olympics be held for the first time after 2050?

8 Yinting wants to start adding weight lifting to her exercise routine. Her trainer suggests that she starts with 40 repetitions for the first week and then increases by six repetitions each week after. Find the number of weeks *before* Yinting will be doing 82 repetitions.

A sequence is **geometric** when each term after the first is found by multiplying by a non-zero number, called the **common ratio (r)**.

The common ratio can be found by taking any term and dividing by the previous term: $r = \dfrac{u_n}{u_{n-1}}$.

When $r < -1$ or $r > 1$, the sequence is **diverging**.

When $-1 < r < -1$, $r \neq 0$, the sequence is **converging**.

The general formula for any geometric sequence can be derived similarly to arithmetic sequences on page 14:

$u_1, u_2, u_3, u_4, u_5, \ldots$

$= u_1, r \times u_1, r \times r \times u_1, r \times r \times r \times u_1, r \times r \times r \times r \times u_1, \ldots$

$= u_1, u_1 r, u_1 r^2, u_1 r^3, u_1 r^4, \ldots$

> Looking at the pattern between the term number and the exponent of r, we can see that the formula for any term in the sequence is
> $$u_n = u_1 r^{n-1}.$$

EXAM HINT

If $r = 1$, then you will have a constant sequence, not a progression, as all the terms will be the same!

Investigation 4

For each of the given conditions below, generate the first five terms of the geometric sequence using a table:

1 $u_1 = 3, r = 2$

2 $u_1 = -4, r = -2$

3 $u_1 = 100, r = \dfrac{1}{2}$

4 $u_1 = -7299, r = -\dfrac{1}{3}$

5 $u_1 = 10\,000, r = \dfrac{1}{10}$

6 $u_1 = 1, r = 3$

A sequence is called **diverging** if the absolute value of each subsequent term is getting larger, tending away from 0.

A sequence is called **converging** if the absolute value of each subsequent term is getting smaller, approaching 0.

7 **Factual** Sort the sequences into two sets: convergent or divergent.

8 **Conceptual** What do you notice about the value of r in each of the sets?

9 **Conceptual** How can you tell whether a sequence is converging or diverging?

Example 10

For each geometric progression, write down the value of r and find the term indicated.

a   40, 20, 10, ... u_{12} **b** $\dfrac{1}{2}, \dfrac{-1}{4}, \dfrac{1}{8}, ... u_9$ **c**   87, 8.7, 0.87, ... u_6 **d** $7x, 14x^2, 21x^3, ... u_8$

a $r = \dfrac{u_n}{u_{n-1}} = \dfrac{20}{40} = \dfrac{1}{2}$ $u_n = u_1 r^{n-1}$ $u_{12} = 40\left(\dfrac{1}{2}\right)^{12-1}$ $u_{12} = 40\left(\dfrac{1}{2}\right)^{11}$ $u_{12} = 40\left(\dfrac{1}{2048}\right)$ $u_{12} = \dfrac{5}{256}$ **b** $r = \dfrac{-\dfrac{1}{4}}{\dfrac{1}{2}} = \dfrac{-1}{2}$ $u_n = u_1 r^{n-1}$ $u_9 = \dfrac{1}{2}\left(-\dfrac{1}{2}\right)^{9-1}$ $u_9 = \dfrac{1}{2}\left(\dfrac{-1}{2}\right)^{8}$ $u_9 = \dfrac{1}{512}$ **c** $r = \dfrac{u_n}{u_{n-1}} = \dfrac{8.7}{87} = 0.10$ $u_n = u_1 r^{n-1}$ $u_6 = 87(0.10)^{6-1}$ $u_6 = 87(0.10)^{5}$ $u_6 = 87(0.00001)$ $u_6 = 0.00087$ **d** $r = \dfrac{u_n}{u_{n-1}} = \dfrac{14x^2}{7x} = 2x$ $u_n = u_1 r^{n-1}$ $u_8 = 7x(2x)^{8-1}$ $u_8 = 7x(2x)^{7}$ $u_6 = 7x(128x^7)$ $u_6 = 896x^8$	First calculate r by dividing any term by the previous term. Substitute u_1, n and r into the general term formula and simplify.

Exercise 1F

1 For the following sequences, determine whether they are geometric and if so, find the term indicated.

   a $9, 27, 81, \ldots u_6$ **b** $2, -12, 18, \ldots u_9$

   c $6, 4.5, 3.375, \ldots u_7$ **d** $-4, 6, -9, \ldots u_8$

   e $500, 100, 20, \ldots u_{13}$ **f** $4, 9, 16, \ldots u_7$

   g $3, 3m, 3m^2, \ldots u_{12}$

2 Suppose you find one cent on the first day of September and two cents on the second day, four cents on the third day, and so on. How much money will you find on the last day of September?

3 Write the first five terms of a geometric sequence in which the sixth term is 64.

Example 11

Given $u_1 = 81$ and $u_6 = \dfrac{1}{729}$ find the values of u_2, u_3, u_4 and u_5 such that the sequence is geometric.

$u_n = u_1 r^{n-1}$ $u_6 = u_1 r^{6-1}$ $\dfrac{1}{729} = 81 r^5$ $r^5 = \dfrac{1}{59\,049}$ $r = \sqrt[5]{\dfrac{1}{59\,049}} = \dfrac{1}{9}$ $u_2 = r \times u_1 = 81\left(\dfrac{1}{9}\right) = 9$ $u_3 = r \times u_2 = 9\left(\dfrac{1}{9}\right) = 1$ $u_4 = r \times u_3 = 1\left(\dfrac{1}{9}\right) = \dfrac{1}{9}$ $u_5 = r \times u_4 = \left(\dfrac{1}{9}\right)\left(\dfrac{1}{9}\right) = \dfrac{1}{81}$ Therefore, the sequence is $81, 9, 1, \dfrac{1}{9}, \dfrac{1}{81}, \dfrac{1}{729}$.	This question is asking you to complete the blanks: $81, \underline{\quad}, \underline{\quad\quad}, \underline{\quad\quad}, \underline{\quad\quad}, \dfrac{1}{729}$ Note: The term between two existing terms is sometimes called "geometric means". This makes $u_1 = 81$ and $u_6 = \dfrac{1}{729}$. In order to find the missing terms, you first need to find r. Now you can find the four missing terms by multiplying each term by r, starting with u_2.

Example 12

In a geometric sequence, $u_1 = 5$ and $u_5 = 1\,280$. The last term of the sequence is $20\,480$. Given that r is positive, find the number of terms in the sequence.

$u_n = u_1 r^{n-1}$ $u_5 = u_1 r^{5-1}$	You first need to find r:

$1280 = 5r^4$	Now you can use $u_1 = 5$, $r = 4$ and $u_n = 20\,480$ to find n.
$r^4 = 256$	
$r = \sqrt[4]{256}$	
$r = 4$	
$u_n = u_1 r^{n-1}$	
$20\,480 = 5 \times 4^{n-1}$	
$4\,096 = 4^{n-1}$	
$4^6 = 4^{n-1}$	
$6 = n - 1$	
$n = 7$	

Example 13

The value of a car depreciates at a rate of 19% each year. If you buy a new car today for $33\,560, how much will it be worth after four years?

$r = 100\% - 19\% = 81\% = 0.81$	Since you want to find the value of the car, you need to subtract the depreciation from 100% to find r.
$u_n = u_1 r^{n-1}$	
$u_5 = 33\,560(0.81)^{5-1}$	Since u_1 is when time = 0, after four years would be the fifth term.
$u_5 = 33\,560(0.81)^4$	
$u_5 = 14\,446.47956\ldots$	Note: Be sure to use approximation signs for any rounded values. Unless otherwise stated, monetary values should be rounded to two decimal places.
The car is worth $\approx \$14\,446.48$	

Exercise 1G

1 If a geometric sequence has $u_5 = 40$ and $u_{10} = 303.75$, find the value of u_{15}.

2 For a geometric sequence that has $r = -\dfrac{4}{5}$ and $u_6 = -1\,280$, find the 20th term.

3 If 16, $x + 2$, 1 are the first three terms of a geometric sequence, find all possible values of x.

4 Find the number of terms in the geometric sequence

6, 12, 24, …, 1 536

5 Find the possible values of the common ratio of a geometric sequence whose first term is 2 and whose fifth term is 32.

6 In 2017, the number of students enrolled in a high school was 232. It is estimated that the student population will increase by 3% every year. Estimate the number of students that will be enrolled in 2027.

7 An old legend states that a peasant won a reward from a king. The peasant asked to be paid in rice; one grain on the first square of a chessboard, two grains on the second, four on the third square, and so on.

a How many grains of rice would be on the 30th square?

b Which square would contain exactly 512 grains of rice?

8 Given $u_1 = 8$ and $u_5 = 128$, find the values of u_2, u_3 and u_4 such that the sequence is geometric.

Investigation 5

Consider each of the sequences below:

a 20, 16, 12, …   **b** 20, 10, 5, …

c −4, −12, −36, …   **d** −16, −18, −20, …

e 1, 2, 6, …   **f** $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, …$

g 2, 4, 6, …   **h** 3, 6, 7, …

1 Decide whether each sequence is geometric, arithmetic or neither.

2 Find the general term for each sequence, if possible.

3 **Factual** What is a common difference? What is a common ratio?

4 **Conceptual** How can you determine whether a sequence is geometric, arithmetic or neither?

5 **Conceptual** How do the common difference or common ratio help you to describe and generalize a sequence?

Exercise 1H

For each question below, first decide whether the situation is arithmetic or geometric, then solve accordingly.

1 A frog fell into a 1 m well and wanted to go back up to the top of the well. Every day it moved up half the distance to the top. After 10 days, how much did the frog have left to climb?

2 Your grandparents deposit $2000 into a bank account to start a college fund for you. They will continue to deposit a fixed amount each month if you deposit $5 a month as well. In 36 months, you would like to have $6500 in the account. How much will they have to contribute each month?

3 The Chinese zodiac associates years with animals, based on a 12-year cycle. Samu was born in 1962, the Year of the Tiger. He lives in Finland, which celebrated its centennial in 2017. Did Finland gain its independence in the Year of the Tiger?

4 A scientist puts six bacteria, which multiply at a constant rate, in a Petri dish. She records the number of bacteria each minute thereafter. If she counts 324 bacteria 20 minutes later, at what rate are the bacteria reproducing?

1.3 Arithmetic and geometric series

Arithmetic series

> ### Investigation 6
>
> Let's make a triangle out of coins. Place one coin on the table.
>
> Underneath it, create a row of two coins.
>
> Place three in the next row.
>
> Continue adding rows, each with one more coin than the row before.
>
> **1** Is the sequence between the row number and number of coins in that row arithmetic or geometric?
>
> **2** Write the general formula for the number of coins in any row.
>
> **3** Complete the table showing the **total number** of coins it will take to make a triangle with the given number of rows:
>
> | Number of rows | Total number of coins |
> |---|---|
> | 1 | 1 |
> | 2 | 3 |
> | 3 | |
> | 4 | |
> | 5 | |
>
> **4** Hypothesize how many coins are needed to create a triangle with 10 rows. How many coins are needed for 20 rows? How many coins are needed for n rows?

The sum of the terms of an arithmetic sequence is called an **arithmetic series**.

An example of an arithmetic series is $3 + 6 + 9 + 12 + \ldots 30$.

Carl Gauss, a German mathematician, was the first to determine the formula to calculate the sum of an arithmetic sequence. When he tried to add up the natural numbers from 1 to 100, he noticed he could match them into pairs with equal sum:

$$1 + 2 + 3 + \cdots 98 + 99 + 100$$

$$1 + 100 = 101 \qquad 2 + 99 = 101 \qquad 3 + 98 = 101$$

Since there would be 50 pairs whose sum is 101, the total sum must be $50 \times 101 = 5050$.

Consider the same process with the general arithmetic series:

$$u_1 + u_2 + u_3 + \dots u_n$$

We can match the pairs into equal sums:

$$u_1 + u_n \qquad u_2 + u_{(n-1)} \qquad u_3 + u_{(n-2)}$$

Since we started with n terms, there would be $\frac{n}{2}$ sums.

> The sum, S_n, would be expressed as $S_n = \frac{n}{2}(u_1 + u_n)$.

Since we already have the formula $u_n = u_1 + (n-1)d$, we can substitute it into the sum formula:

$$S_n = \frac{n}{2}(u_1 + u_n) = \frac{n}{2}(u_1 + u_1 + (n-1)d) = \frac{n}{2}(2u_1 + (n-1)d)$$

> This gives us a second formula for the sum:
> $$S_n = \frac{n}{2}\left[2u_1 + (n-1)d\right]$$

Geometric series

> For geometric series, $u_1 + u_1 r + u_1 r^2 + \cdots u_1 r^{n-2} + u_1 r^{n-1}$, the formula for the sum is:
> $$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}, r \neq 1$$

Example 14

For each infinite series below, decide whether it is arithmetic or geometric, then find the sum to the given term:

a $1 + 5 + 9 + \dots u_8$

b $6 + 12 + 18 + \dots u_7$

c $-2 - 4 - 8 - \dots u_9$

d $100 + 50 + 25 + \dots u_6$

e $132 + 124 + 116 + \dots u_7$

f $\frac{1}{8} - \frac{1}{4} + \frac{1}{2} - \dots u_{12}$

The Towers of Hanoi

The problem

The aim of the **Towers of Hanoi problem** is to move all the disks from peg A to peg C following these rules:

1 Move only one disk at a time.

2 A larger disk may not be placed on top of a smaller disk.

3 All disks, except the one being moved, must be on a peg.

For 64 disks, what is the minimum number of moves needed to complete the problem?

Explore the problem

Use an online simulation to explore the Towers of Hanoi problem for three and four disks.

What is the minimum number of moves needed in each case?

Solving the problem for 64 disks would be very time consuming, so you need to look for a rule for n disks that you can then apply to the problem with 64 disks.

Try and test a rule

Assume the minimum number of moves follows an arithmetic sequence.

Use the minimum number of moves for three and four disks to predict the minimum number of moves for five disks.

Check your prediction using the simulator.

Does the minimum number of moves follow an arithmetic sequence?

Find more results

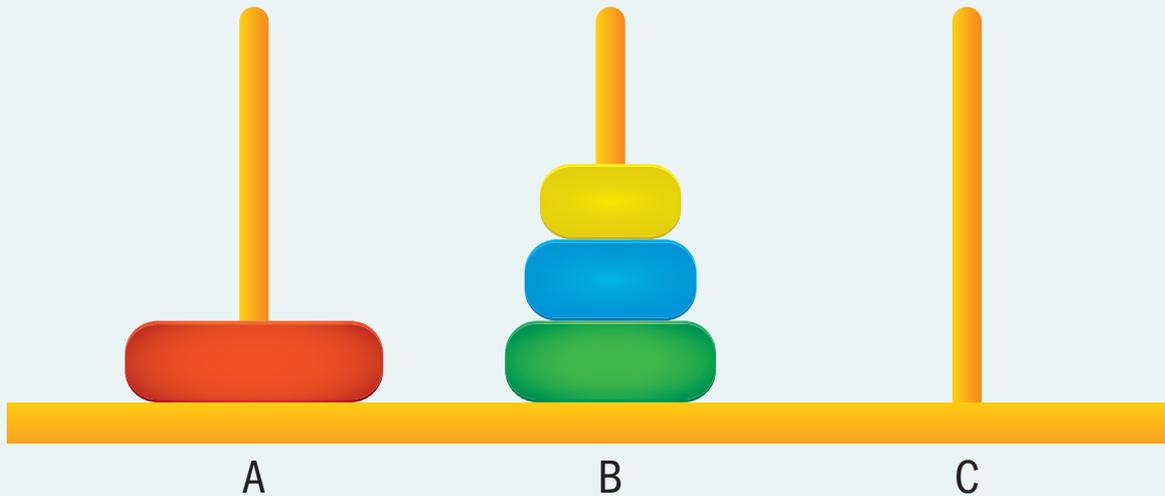Use the simulator to write down the number of moves when $n = 1$ and $n = 2$.

Organize your results so far in a table.

Look for a pattern. If necessary, extend your table to more values of n.

Try a formula

Return to the problem with four disks.

Consider this image of a partial solution to the problem. The large disk on peg A has not yet been moved.

A   B   C

Consider your previous answers.

What is the minimum possible number of moves made so far?

How many moves would it then take to move the largest disk from peg A to peg C?

When the large disk is on peg C, how many moves would it then take to move the three smaller disks from peg B to peg C?

How many total moves are therefore needed to complete this puzzle?

Use your answers to these questions to write a formula for the minimum number of moves needed to complete this puzzle with n disks.

This is an example of a **recursive formula**. What does that mean?

How can you check if your recursive formula works?

What is the problem with a recursive formula?

Try another formula

You can also try to solve the problem by finding an **explicit formula** that does not depend on you already knowing the previous minimum number.

You already know that the relationship is not arithmetic.

How can you tell that the relationship is not geometric?

Look for a pattern for the minimum number of moves
in the table you constructed previously.

Hence write down a formula for the minimum number of moves in terms of n.

Use your explicit formula to solve the problem with 64 disks.

Extension

- What would a solution look like for four pegs? Does the problem become harder or easier?
- Research the "Bicolor" and "Magnetic" versions of the Towers of Hanoi puzzle.
- Can you find an explicit formula for other recursive formulae? (eg Fibonacci)