OXFORD IB PREPARED

MATHEMATICS: APPLICATIONS AND INTERPRETATION

IB PREPARED

IB DIPLOMA PROGRAMME

OXFORD

Contents

Worked solutions to end-of-chapter practice questions and exam papers in this book can be found on your free support website. Access the support website here:
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5 CALCULUS

5.1 DIFFERENTIATION

You should know:

- ✔ the informal concept of a limit
- ✔ the derivative of a function f gives the rate of change of the dependent variable with respect to the independent variable. This is equivalent to the gradient of the curve $y = f(x)$
- ✔ forms of notation for the first derivative, for example: $\frac{dy}{dx}$, $f'(x)$, $\frac{dV}{dt}$
- ✔ for increasing functions $f'(x) > 0$, and for decreasing functions $f'(x) < 0$
- ✔ the derivative of $f(x) = ax^n$ is $f'(x) = ax^{n-1}$, $n \in \mathbb{Z}$
- ✔ local maximum and minimum points occur where $f'(x) = 0$, but these might not be the greatest or least values in the given domain.

You should be able to:

- ✔ find derivatives of functions of the form $f(x) = ax^n + bx^{n-1} + ...$ where all exponents are integers
- ✔ find the equations of tangents and normals at a given point, both analytically and using technology
- ✔ use technology to find values of $f'(x)$ given $f(x)$, and find the solutions of $f'(x) = 0$
- ✔ solve optimization problems in context.

Note

In the case of an equation like $P = 2t^2 + 3t - 7$ the notation is $\frac{dP}{dt} = 4t + 3$

The derivative of a function at a point gives the **gradient** of the graph of the function at that point.

The derivative of a function or the gradient of a curve gives the **rate of change** of the function. The larger the value the greater the rate of change.

A positive value for the derivative means the value of the dependent variable is increasing as the independent variable increases and a negative value means it is decreasing.

The derivative can be written as $\frac{dy}{dx}$ or $f'(x)$

The derivative of a polynomial can be found using the relation:

$y = ax^n \Rightarrow \frac{dy}{dx} = anx^{n-1}$ on each of the terms.

Note the special case of $y = mx$ and $y = c$ which have derivatives $\frac{dy}{dx} = m$ and $\frac{dy}{dx} = 0$ respectively.

SAMPLE STUDENT ANSWER

Consider the function $f(x) = x^3 - 3x^2 + 2x + 2$

(a) Find $f'(x)$

(b) There are two points at which the gradient of the graph of f is 11. Find the x-coordinates of these points.

(a) $f(x) = x^3 - 3x^2 + 2x^1 + 2$

$f'(x) = 3x^2 - 6x^1 + 2$

$f'(x) = 3x^2 - 6x + 2$

(b) $11 = 3x^2 - 6x + 2$

$3x^2 - 6x + 2 = 11$

$3x^2 - 6x - 9 = 0$

GDC = Poly roots

$\{-1, 3\}$

(1) $f(-1) = (-1)^3 - 3(-1)^2 + 2(-1) + 2$

$= -4$

(2) $f(3) = 3^3 - 3(3)^2 + 2(3) + 2$

$= 8$

Answers:

(a) $f'(x) = 3x^2 - 6x + 2$

(b) $(-1, -4)$, $(3, 8)$

▲ Gradient equal to 11 means $f'(x) = 11$

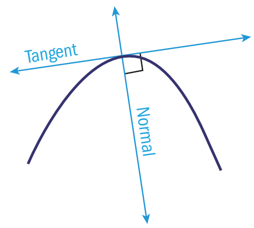
▼ This could be solved by factorising, but always take the easiest route in an exam

▲ Clearly set out.

▼ Unfortunately the question just asked for x-coordinates. The student probably did not lose any marks because they found the x-coordinates, but time was lost unnecessarily in calculating the y-coordinates. Always read the question carefully.

Tangents and normal

The tangent to a curve at a point is the line which just touches the curve at that point and has the same gradient as the curve at that point. The normal to the curve at a point is the line through the point, perpendicular to the tangent.



Remember from section 3.3, that if a line has a gradient of m, then the line perpendicular to it has a gradient of $-\frac{1}{m}$

Example 5.1.1

Consider the curve $y = \frac{x^2}{4}$

(a) Find $\frac{dy}{dx}$

(b) Find the equation of the tangent to the graph of $y = \frac{x^2}{4}$ at $x = 4$

(c) Find the x-coordinate of the point at which the normal to the graph of $y = \frac{x^2}{4}$ has gradient $-\frac{1}{8}$

Solution

(a) $\frac{dy}{dx} = \frac{2x}{4} = \frac{x}{2}$

$\frac{x^2}{4}$ can be written as $\frac{1}{4}x^2$ so the derivative is $\frac{1}{4} \times 2x = \frac{1}{2}x$ or $\frac{x}{2}$

(b) When $x = 4$, $\frac{dy}{dx} = \frac{4}{2} = 2$

Equation is $y = 2x + c$

The equation of the tangent is $y = mx + c$

The first stage is to find the value of m, which is the gradient of the tangent and is equal to the gradient of the curve at $x = 4$

When $x = 4$, $y = \frac{16}{4} = 4$

so $4 = 2 \times 4 + c \Rightarrow c = -4$

$y = 2x - 4$

To find the value of c a point on the line is needed, so we can use the point where the tangent meets the curve, (4, 4)

An alternative method is to use $y - y_1 = m(x - x_1)$

(c) Gradient of tangent = 8

$\frac{x}{2} = 8 \Rightarrow x = 16$

If the gradient of a tangent is equal to m then the gradient of the normal is equal to $-\frac{1}{m}$ and vice versa.

Note

All GDCs will be able to find the numerical value of a derivative at a point, and some will also be able to find the equation of a tangent at point.

Optimization

If $f'(x) > 0$ then the function f is increasing, and if $f'(x) < 0$ the function is decreasing.

When $f'(x) = 0$ the curve will have a maximum or minimum point. This fact can be used to solve **optimization** problems.

Note

Optimization is 'the action of making the best or most effective use of a situation or resource'. In this topic it is finding the maximum or minimum value of a function in a given context.

Note

In the HL course, $f'(x) = 0$ can also occur at a horizontal point of inflection.

SAMPLE STUDENT ANSWER

A potter sells x vases per month.

His monthly profit in Australian dollars (AUD) can be modelled by

$$P(x) = -\frac{1}{5}x^3 + 7x^2 - 120,\ x \geq 0$$

(a) Find the value of P if no vases are sold.

(b) Find $P'(x)$

(c) Hence find the number of vases that will maximize the profit.

(a) $P(0) = -\frac{1}{5}(0)^3 + 7(0)^2 - 120$

$= -120$

(b) $P(x) = -\frac{1}{5}x^3 + 7x^2 - 120$

$= -\frac{3}{5}x^2 + 14x$

(c) $0 = -\frac{3}{5}x^2 + 14x$

$x = \frac{70}{3}$

$= 23.3$

▼ The answer is correct but the candidate is in danger of losing a mark by not beginning the second line with $P'(x) =$

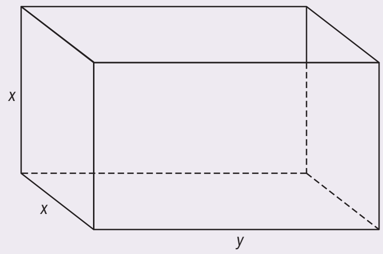
▲ The maximum of the curve occurs when $P'(x) = 0$. It is important to show the full working to indicate to the examiner that the maximum has not been found directly from the GDC.

▼ The number of vases must be a whole number, so the answer needs to be rounded. Always keep the context of the question in mind as you are answering it.

In some optimization questions, the function to be optimized will contain two variables. In this case the question will provide information about a second relationship between the two variables (a constraint) which will allow you to substitute out one of the variables.

Example 5.1.2

Fred makes an open metal container in the shape of a cuboid, as shown here.



The container has a height x m, width x m and length y m. The volume is 36 m^3.

Let $A(x)$ be the outside surface area of the container.

(a) Show that $A(x)=\frac{108}{x}+2x^2$

(b) Find $A'(x)$

(c) Hence find the height of the container which minimizes the surface area.

Solution

(a) $A(x)=2x^2+3xy$

$V=36=x^2y\Rightarrow y=\frac{36}{x^2}$

There are two square sides and three rectangular sides, as the container is 'open'.

$A(x)=2x^2+3x\times\frac{36}{x^2}$

$=2x^2+\frac{108}{x}$

The equation has two variables so we need to use the condition that the volume is 36 m^3 to find an expression for y, which can then be substituted into $A(x)$.

(b) $A(x)=2x^2+108x^{-1}$

$A'(x)=4x-108x^{-2}$

First write using negative exponents.

(c) $4x-\frac{108}{x^2}=0$

$4x^3=108\Rightarrow x^3=27$

$x=3$

When manipulating an expression it is often easiest to convert a negative exponent to a fraction.

Height is 3 m.

Because the question said 'hence' you need to make it clear that you are using the answer to part (b).

Having written the first line, this equation could then be solved using the GDC, rather than analytically.

If expressions are given as fractions, they should be written with negative exponents before differentiating. For example:

Rewrite $y=\frac{2}{x^2}$ as

$y=2x^{-2}\Rightarrow\frac{dy}{dx}=-4x^{-3}$

Remember that if the domain is restricted then the maximum and minimum values might occur at the end points rather than where the derivative is equal to zero. In these cases you should always check by plotting the curve on your GDC.

5.2 INTEGRATION

You should know:

- ✔ if $\frac{dy}{dx} = ax^n$ then $y = \frac{a}{n+1}x^{n+1} + c$, for $n \neq -1$, which can be written as
  $\int ax^n \, dx = \frac{a}{n+1}x^{n+1} + c$
- ✔ the area of a region enclosed by a curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$, where $f(x) > 0$, can be found from calculating $\int_a^b f(x)dx$
- ✔ the trapezoidal rule.

You should be able to:

- ✔ find the general form of a function when given its derivative or rate of change
- ✔ find the value of c using a **boundary condition,** for example the value of y when x is 0
- ✔ use technology to find the area of a region enclosed by a curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$, where $f(x) > 0$
- ✔ find an estimate for the value of an area using the trapezoidal rule, with intervals of equal width when given either a table of data or a function.

Antiderivatives

An antiderivative or integral is useful for deriving an equation from a rate or for finding areas under a curve.

If $\frac{dy}{dx} = ax^n$ then $y = \frac{a}{n+1}x^{n+1} + c$, $n \neq -1$

Note

This can be written as

$\int ax^n \, dx = \frac{a}{n+1}x^{n+1} + c$

This formula can be found in section SL 5.5 of the formula book. Notice that it excludes the case where $n = -1$ as the formula in this instance would involve dividing by 0.

Assessment tip

In exams, n will always be an integer.

Example 5.2.1

The rate at which the temperature T is changing t minutes after a heating element is turned on is given by the equation,

$\frac{dT}{dt} = 19 - 2t$, $0 \leq t \leq 10$

(a) Find the rate of change of the temperature when $t = 4$

When $t = 0$ the temperature is 5°C

(b) Find an expression for the temperature at time t

(c) Find the maximum value of T for $0 \leq t \leq 10$

Solution

(a) When $t = 4$

$\frac{dT}{dt} = 19 - 8 = 11$

11 °C per minute

The rate of change is another way of asking for the value of the derivative.

Notice that the units of $\frac{dT}{dt}$ are °C per minute

(b) $T = \int 19 - 2t \, dt$

$= 19t - 2 \times \frac{1}{2}t^2 + c$

$= 19t - t^2 + c$

Note that $\int 19 dt = 19t$ because the derivative of $19t$ is 19.

The value of c is found by substituting known values for the variables.

When $t = 0$,
$T = 19 \times 0 - 0^2 + c = 5$

$\Rightarrow c = 5$

$T = 19t - t^2 + 5$

(c) The maximum value occurs when $t = 9.5$

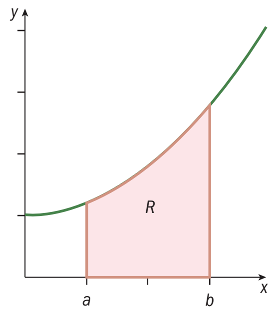
$T = 95.25\ ^\circ\text{C}$

You should plot the curve on your GDC to ensure the maximum does not occur at $t = 0$ or $t = 10$. Having plotted the curve it is easier to find the maximum directly from the curve rather than solving $\frac{dT}{dt} = 0$

Areas between a curve and the *x*-axis

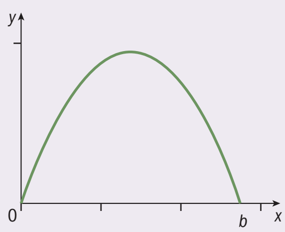
Integrals can be used to find the area between a curve and the x-axis.

The notation for the area between the curve $y = f(x)$, the lines $x = a$ and $x = b$ where $b > a$ (shown as R in the diagram) is given by the **definite integral** written as $\int_a^b f(x)\mathrm{d}x$



Example 5.2.2

The side wall in a concert hall can be modelled by a curve with equation $y = 2.8x - 0.5x^2$, $0 \le x \le b$ where x is the horizontal distance from the point O and the units are measured in metres.



(a) Find the value of b.

(b) Write down an integral which represents the area of the wall.

(c) It is intended to repaint the wall. If one can of paint covers $4.5\,\text{m}^2$ of wall, find the minimum number of cans of paint needed.

Solution

(a) $b = 5.6$

(b) $\int_0^{5.6} 2.8x - 0.5x^2\,\mathrm{d}x$

This can be found by factorising the equation, or directly from the GDC by entering the equation and finding the zero (x-intercept).

(c) Area $= 14.6\ \text{m}^2$

Using the integral formula for area.

Number of cans required $\frac{14.6}{4.5} \approx 3.24$

The area is obtained directly from the GDC.

4 cans of paint

To find the minimum number of cans you need to round up.

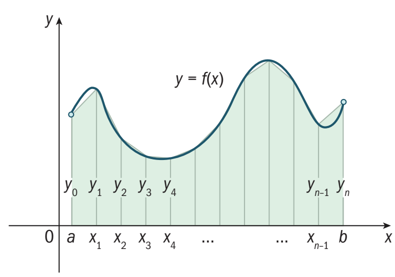
Assessment tip

At Higher Level you will need to know how to work out this value using integration (anti-derivatives) but for Standard Level you will always use the appropriate function on the calculator. You should though know how to use the notation correctly.

The trapezoidal rule

Before the arrival of GDCs, an approximation for the area between a curve and the x-axis was often found by dividing the area into trapezoids and working out their areas. This is still useful today, particularly when the equation defining the curve is not known.

The area between the curve $y = f(x)$, the lines $x = a$ and $x = b$ with $b > a$, can divided into n trapezoids each with height $h = \frac{b-a}{n}$, as shown.



The area of the trapezoids can be found using the trapezoidal rule:

$$\int_a^b f(x)\mathrm{d}x \approx \frac{1}{2}h\left(y_0 + y_n + 2(y_1 + y_2 + \ldots + y_{n-1})\right)$$

This formula is in section SL 5.8 of the formula book

The question may ask you if the trapezium rule gives an underestimate or an overestimate of the actual area. This is usually clear from the diagram.

Example 5.2.3

Use the trapezoidal rule with 4 trapezoids to find an approximate value for the area between the curve $y = 2 + \frac{6}{x}$, the x-axis and the lines $x = 1$ and $x = 3$

Solution

$h = \frac{3-1}{4} = 0.5$

Using the formula $h = \frac{b-a}{n}$

x	1	1.5	2	2.5	3
y	8	6	5	4.4	4

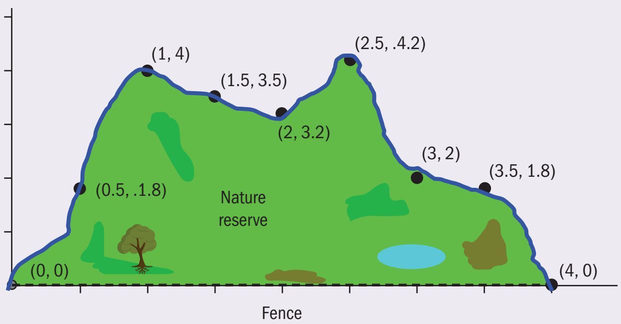
Area $\approx \frac{1}{2} \times 0.5((8+4) + 2(6+5+4.4))$

$= 10.7$

You should enter the function into your GDC and read off the required values. Alternatively they can be worked out individually. For example, when $x = 1$, $y = 2 + 6 = 8$. You should always write down the y-values to obtain method marks in case you make an error in your calculations.

Example 5.2.4

A nature reserve is bounded by a river and a straight fence, as shown in the diagram below. Use the trapezoidal rule and the coordinates of the points shown to find the approximate area of the nature reserve. Each unit is one kilometre.



Solution

Width of trapezoids $= 0.5$

Area $\approx \frac{1}{2} \times 0.5((0+0) + 2(1.8 + 4 + 3.5 + 3.2 + 4.2 + 2 + 1.8)) = 10.25\ \text{km}^2$

The width can easily be found from the difference between the x-values, but could also be calculated using $h = \frac{4-0}{8} = 0.5$

5.3 DIFFERENTIATION (HL)

You should know

- ✔ the derivatives of $\sin x$, $\cos x$, $\tan x$, e^x, $\ln x$, x^n where $n \in \mathbb{Q}$
- ✔ if $f''(x) < 0$ the curve is concave down and if $f''(x) > 0$ the curve is concave up
- ✔ that a point of inflection is a point at which the concavity changes and interpretation of this in context.

You should be able to

- ✔ use the chain rule, product rule and quotient rules
- ✔ connect variables with related rates of change
- ✔ use the second derivative to distinguish between local maximum and local minimum points.

At higher level you need to be aware of the following derivatives:

$f(x)$	$\sin x$	$\cos x$	$\tan x$	e^x	$\ln x$	$x^n, n \in \mathbb{Q}$
$f'(x)$	$\cos x$	$-\sin x$	$\frac{1}{\cos^2 x}$	e^x	$\frac{1}{x}$	nx^{n-1}

These are given in section 5.9 and 5.11 of the formula book.

The chain rule

The chain rule is used to differentiate composite functions. For example if u is a function of x, and $y = g(u)$ then $\frac{dy}{dx} = g'(u) \times u'(x)$ or $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Note

This can be remembered by regarding the terms as fractions and 'cancelling' du

The way this is done in practice is to separate the two functions, differentiate both and then multiply them.

So if $y = \sin(x^2)$ then the two functions are $u = x^2$ and $y = \sin u$. These are differentiated and multiplied together:

$$\frac{dy}{dx} = 2x \times \cos u = 2x \cos x^2$$

This formula is in section 5.9 of the formula book.

Try differentiating the functions in the next example before checking the answers.

Assessment tip

Make sure your GDC is in radian mode when the question involves calculus and trigonometric functions.

Example 5.3.1

Differentiate the following functions.

(a) $y = (3x+1)^{\frac{1}{2}}$ (b) $y = e^{\cos x}$ (c) $f(x) = \sin^2 x$

Solution

(a) $u = 3x+1,\ y = u^{\frac{1}{2}}$ — First identify the two functions.

$\frac{du}{dx} = 3,\ \frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$ — Differentiate each one.

$\frac{dy}{dx} = 3 \times \frac{1}{2}u^{-\frac{1}{2}} = \frac{3}{2}(3x+1)^{-\frac{1}{2}}$ — Find the product, changing all variables back to x.

(b) $u = \cos x,\ y = e^u$ — u is often the function in brackets, but sometimes the brackets are not shown so you need to consider which is the first function performed.

$\frac{du}{dx} = -\sin x,\ \frac{dy}{du} = e^u$

$\frac{dy}{dx} = -\sin x \times e^u = -(\sin x)e^{\cos x}$